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CONTENTS

		Page
Pramila Srivastava	On the second theorem of consistency for strong Riesz summability	1
R. N. Sen	A note on symmetric affine connections	17
B. N. Prasad and J. A. Siddiqi	On the Gibbs phenomenon for the Nörlund means	21
Shri Nivas Bhatt	A Tauberian theorem for absolute Riesz summability	29
G. D. Dikshit	On the absolute Riesz summability factors of infinite series (I) ...	33
T. Pati and S. R. Sinha	On the absolute summability factors of Fourier series	41

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ON THE SECOND THEOREM OF CONSISTENCY FOR STRONG RIESZ SUMMABILITY

By

PRAMILA SRIVASTAVA

(*Received November* 25, 1958)

1.1. Let Σa_n be a given infinite series, and λ_n a positive monotonic increasing function of n, tending to infinity with n. For $r>-1$, we write

$$A_\lambda{}^r(\omega)=\sum_{\lambda_n<\omega}(\omega-\lambda_n)^r a_n.$$

We also write

$$A_\lambda{}^0(\omega)=A_\lambda(\omega).$$

When $k\geqslant 0$, the series Σa_n is said to be summable (R, λ, k) to the sum s, if

$$\lim_{\omega\to\infty}\omega^{-k}A_\lambda{}^k(\omega)=s.$$[1]

The series Σa_n is said to be strongly summable (R, λ, k), or summable $[R, \lambda, k]$, to the sum s, where $k>0$, if

$$\int^\omega |x^{-(k-1)}A_\lambda{}^{k-1}(x)-s|\,dx=o(\omega),$$

as $\omega\to\infty$.[2]

1.2. The classical second theorem of consistency due to Hardy and Riesz[3] is to the effect that if a series Σa_n is summable (R, λ, k), where $\lambda_n=e^{\mu_n}$, then it is also summable (R, μ, k) to the same sum. Later Hardy[4] generalized this theorem and proved that if the series Σa_n is summable (R, λ, k) and μ is a logarithmico-exponential function of λ, tending to infinity with λ, such that $\mu=O(\lambda^\Delta)$, where Δ is a constant however large, then it is summable (R, μ, k). Finally Hirst[5] removed the limitation on μ of being a logarithmico-exponential function of λ, and proved the following theorem.

1. Hardy and Riesz (5).

2. Srivastava (11); see also Richert (10). The lower limit of the integral may be any finite positive number. Here and elsewhere, in such cases, we omit the lower limit.

3. Hardy and Riesz (5), Theorem 17.

4. Hardy (3).

5. Hirst (6).

If Σa_n is summable (R, λ, k), then it is summable (R, μ, k) to the same sum, where $\mu=\phi(\lambda)$, and $\phi(t)$ is a function which increases steadily to infinity with t and satisfies the following conditions:

(i) *k is an integer*

$$\text{(a)} \quad \int^x t^r |\phi^{(r+1)}(t)|\, dt=O\{\phi(x)\},\ r=1, 2,\ldots,k.$$

(ii) *k is not an integer*

$$\text{(a)} \quad \int^x t^r |\phi^{(r+1)}(t)|\, dt=O\{\phi(x)\},\ r=1, 2,\ldots,h+1,$$

where h is the greatest integer less than k, and either

(b) *$\phi'(t)$ is a monotonic increasing function,*

or (c_1) *$\phi'(t)$ is a monotonic decreasing function*

and (c_2) $t\phi''(t)=O\{\phi'(t)\}$.

Recently it has been shown by Kuttner[1] that the assertion of conditions (a) in the above for $r=1,2,\ldots,k-1$ in the case of integral k and for $r=1,2,\ldots,h$ in the case of non-integral k is redundant.

In Theorem 1 of this paper we establish for strong Riesz summability a consistency theorem which may be looked upon as the direct analogue of Hirst's theorem quoted above. But there seems to be a marked difference in the nature of the consistency theorem for ordinary summability and that for strong summability. We find that when k is non-integral, in Hirst's theorem the condition of taking $\phi'(t)$ to be a monotonic decreasing function, together with $t\phi''(t)=O\{\phi'(t)\}$ serves the purpose for all positive values of k, whereas in our consistency theorem for strong summability this condition on $\phi'(t)$ is effective only as long as k remains greater than one. Because when $k<1$, we have proved in Theorem 2 by giving a *gegenbeispiel* that the consistency theorem breaks down.

My warmest thanks are due to Prof. B. N. Prasad for his kind help and guidance in the preparation of this paper.

2.1. We establish the following theorems.

THEOREM 1. *If Σa_n is summable $[R, \lambda, k]$, then it is summable $[R, \mu, k]$ to the same sum, where $\mu=\phi(\lambda)$, and $\phi(t)$ is a function which increases steadily to infinity with t, and is a $(\kappa+1)$th indefinite integral for $t\geqslant 0$, such that*

(i) *k is an integer: $\kappa=k$,*

$$\text{(a)} \quad \int^\omega t^k |\phi^{(k+1)}(t)|\, dt=O\{\phi(\omega)\};$$

1. Kuttner (9).

(ii) *k is not an integer* : $\kappa=h+1$,

$$\text{(a)} \quad \int^{\omega} t^{h+1} |\phi^{(h+2)}(t)|\, dt=O\{\phi(\omega)\},$$

where h is the greatest integer less than k and either

(b) $\phi'(t)$ *is a monotonic increasing function,*

or (c) $\phi'(t)$ *is a monotonic decreasing function,* $t\phi''(t)=O\{\phi'(t)\}$ *and* $k>1$.

THEOREM 2. *For* $0<k<1$, *there is a series* Σa_n *which is summable* $[R, \lambda, k]$ *but is not summable* $[R, \log \lambda, k]$.

3. We require a number of lemmas for the proof of our theorems.

LEMMA 1.[1] *When k is an integer,*

$$A_{\lambda}(t)=\frac{1}{k!}\left(\frac{d}{dt}\right)^{k} A_{\lambda}{}^{k}(t).$$

LEMMA 2.[2] *If k is nonintegral and h denotes the greatest integer less than k, then*

$$A_{\lambda}{}^{h}(x)=\frac{\Gamma(h+1)}{\Gamma(k+1)\Gamma(h+1-k)}\int_{0}^{x}(x-t)^{h-k} A_{\lambda}{}^{k-1}(t)dt$$

LEMMA 3. *If*

$$\int^{\omega} t^{n} |\phi^{(n+1)}(t)|dt=O\{\phi(\omega)\},$$

then

$$\int^{\omega} t^{r} |\phi^{(r+1)}(t)|dt=O\{\phi(\omega)\},$$

for $r=1, 2, \ldots, n-1$, *and*

$$\omega^{r}\phi^{(r)}(\omega)=O\{\phi(\omega)\},$$

for $r=1, 2, \ldots, n$.

The first part of the lemma has been given by Kuttner[3] and the second part is due to Hirst.[4]

LEMMA 4.[5] *The nth derivative of* $\{F(x)\}^{m}$ *is the sum of a number of terms of the form*

$$A\{F(x)\}^{m-r}\{F^{(1)}(x)\}^{\alpha_1}\ldots\{F^{(n)}(x)\}^{\alpha_n},$$

1. Hardy and Riesz (5).
2. Hardy and Riesz (5), Lemma 6.
3. Kuttner (9).
4. Hirst (6), Lemma 3.
5. de la Vallée Poussin (14).

where A is a constant, $r \leqslant n$ and the α's are positive integers or zeros such that

$$\sum_{\nu=1}^{n} \alpha_\nu = r, \quad \sum_{\nu=1}^{n} \nu\alpha_\nu = n.$$

Further, if m is a positive integer, then $0 < r \leqslant m$.

LEMMA 5. (i) *If*

$$G_1(x) = \int_a^x \phi_1(x, u) g_1(u) du,$$

then

$$\int_a^\omega |dG_1(x)| \leqslant \overline{\underset{a<u}{bd}} \left\{ \phi_1(u, u) + \int_u^\omega |d_x \phi_1(x, u)| \right\} \int_a^\omega |g_1(u)| \, du,$$

and (ii) *if*

$$G_2(x) = \int_0^1 \phi_2(x, u) g_2(u) du,$$

then

$$\int_a^\omega |dG_2(x)| < \overline{\underset{0<u<1}{bd}} \int_a^\omega |d_x \phi_2(x, u)| \int_0^1 |g_2(u)| du.$$

This is similar to Lemma 1 of Tatchell[1] and can be similarly proved.

LEMMA 6. *If $\phi(t)$ is a monotonic increasing function of t and $\phi'(t)$ is monotonic, then*

$$F(x) = \frac{\phi(x) - \phi\{x\zeta + (1-\zeta)s\}}{(x-s)(1-\zeta)},$$

where s ans ζ are fixed, $s < x$ and $0 < \zeta < 1$, is a monotonic function of x, which increases or decreases according as $\phi'(t)$ increases or decreases.

Proof. We have

$$\begin{aligned}(x-s)^2(1-\zeta)F'(x) &= (x-s)[\phi'(x) - \zeta\phi'\{x\zeta + (1-\zeta)s\}] - [\phi(x) - \phi\{x\zeta + (1-\zeta)s\}] \\ &= (x-s)[\phi'(x) - \zeta\phi'\{x\zeta + (1-\zeta)s\}] - (x-s)(1-\zeta)\phi'(\xi),\end{aligned}$$

where $x\zeta + (1-\zeta)s \leqslant \xi \leqslant x$. Hence, when $\phi'(x)$ is increasing,

$$(x-s)^2(1-\zeta)F'(x) \geqslant (x-s)(1-\zeta)\{\phi'(x) - \phi'(\xi)\} > 0.$$

And this shows that $F(x)$ is monotonic increasing. Also when $\phi'(x)$ is decreasing, we have

$$(x-s)^2(1-\zeta)F'(x) \leqslant (x-s)(1-\zeta)\{\phi'(x) - \phi'(\xi)\} < 0,$$

from which it follows that $F(x)$ is a decreasing function of x.

1. Tatchell (13).

LEMMA 7. *If $U(x)$ is a positive monotonic increasing function of x and $V(x)$ is a positive monotonic decreasing function of x, then*

$$\text{(i)} \int_s^\omega |d\{U(x)V(x)\}| \leqslant [U(x)V(x)]_{x=s} + [U(x)V(x)]_{x=\omega} - 2\int_s^\omega U(x)V'(x)dx,$$

$$\text{(ii)} \int_s^\omega |d\{U(x)V(x)\}| \leqslant [U(x)V(x)]_{x=s} + [U(x)V(x)]_{x=\omega} + 2\int_s^\omega U'(x)V(x)dx.$$

The results of this lemma are obvious.

LEMMA 8.[1] *If $f(x)$ is a positive decreasing function of x and $xf'(x) = O\{f(x)\}$, then there exists a constant $c > 1$ such that*

$$f(cx)/f(x) > \tfrac{1}{2}$$

for all $x > x_0$.

4.1. *Proof of Theorem* 1. We suppose, without any loss of generality, the sum s of the given series to be zero. Then, the hypothesis reduces to the condition

$$\int^\omega |A_\lambda^{k-1}(x)|dx = o(\omega^k). \tag{4.1.1}$$

And for the summability $[R, \mu, k]$ of Σa_n to the sum zero, we must have

$$\int^\omega |A_\mu^{k-1}(x)|dx = o(\omega^k). \tag{4.1.2}$$

Case (i) : *k is an integer.* Putting $x = \phi(u)$ and $\omega = \phi(w)$, the required condition (4.1.2) reduces to

$$\int^w |A_\mu^{k-1}\{\phi(u)\}|\phi'(u)du = o\,[\{\phi(w)\}^k]. \tag{4.1.3.}$$

Now, for $k=1$,

$$A_\mu^{k-1}\{\phi(u)\} = A_\mu\{\phi(u)\} = A_\lambda(u), \tag{4.1.4}$$

and, for $k > 1$,

$$A_\mu^{k-1}\{\phi(u)\} = (k-1)\int_{\lambda_0}^u A_\lambda(t)\{\phi(u)-\phi(t)\}^{k-2}\phi'(t)\,dt.$$

Hence, by Lemma 1,

$$A_\mu^{k-1}\{\phi(u)\} = -\frac{1}{(k-1)!}\int_{\lambda_0}^u \left(\frac{d}{dt}\right)^{k-1}\{A_\lambda^{k-1}(t)\}\frac{d}{dt}\{\phi(u)-\phi(t)\}^{k-1}dt.$$

We integrate the right hand side $(k-1)$ times by parts. All the integrated terms vanish except the one arising from the last integration, and we obtain

$$A_\mu^{k-1}\{\phi(u)\} = A_\lambda^{k-1}(u)\{\phi'(u)\}^{k-1} + \frac{(-1)^k}{(k-1)!}\int_{\lambda_0}^u A_\lambda^{k-1}(t)\left(\frac{d}{dt}\right)^k\{\phi(u)-\phi(t)\}^{k-1}dt. \tag{4.1.5}$$

---

1. Hirst (6), Lemma E.

(4.1.4) shows that the above is valid also for $k=1$. From (4.1.3) and (4.1.5) it follows that (4.1.2) will be satisfied if

(4.1.6) $$\int^{w} |A_{\lambda}^{k-1}(u)|\{\phi'(u)\}^{k} du = o[\{\phi(w)\}^{k}]$$

and

(4.1.7) $$\int^{w} \phi'(u)\, du \left| \int_{\lambda_0}^{u} A_{\lambda}^{k-1}(t) \left(\frac{d}{dt}\right)^{k} \{\phi(u)-\phi(t)\}^{k-1} dt \right| = o[\{\phi(w)\}^{k}].$$

Proof of (4.1.6). Integrating by parts and using (4.1.1), we obtain

$$\int^{w} |A_{\lambda}^{k-1}(u)|\{\phi'(u)\}^{k} du = [o(u^{k})\{\phi'(u)\}^{k}]^{w} + o\int^{w} u^{k}\{\phi'(u)\}^{k-1} |\phi''(u)|\, du.$$

Hence, by hypothesis (a) and Lemma 3, since $k \geqslant 1$, we get

$$\int^{w} |A_{\lambda}^{k-1}(u)|\{\phi'(u)\}^{k} du = o[\{\phi(w)\}^{k}] + o[\{\phi(w)\}^{k-1}] \int^{w} u\,|\phi''(u)|\, du$$
$$= o[\{\phi(w)\}^{k}].$$

Proof of (4.1.7). By Lemma 4,

$$\int_{\lambda_0}^{u} A_{\lambda}^{k-1}(t) \left(\frac{d}{dt}\right)^{k} \{\phi(u)-\phi(t)\}^{k-1} dt$$

is the sum of constant multiples of the integrals of the form

$$I_r(u) \equiv \int_{\lambda_0}^{u} A_{\lambda}^{k-1}(t) \{\phi(u)-\phi(t)\}^{k-1-r} \Pi\{\phi^{(n)}(t)\}^{\alpha_n} dt,$$

where $\Sigma\alpha_n = r$, $\Sigma n\alpha_n = k$, $r \leqslant k-1$. Now integrating $I_r(u)$ by parts and using (4.1.1), we obtain

$$I_r(u) = \Big[o(t^{k}) \Pi\{\phi^{(n)}(t)\}^{\alpha_n}\Big]_{r=k-1}^{u}$$
$$+ o\left(\int_{\lambda_0}^{u} t^{k} \left| \frac{d}{dt} \{\phi(u)-\phi(t)\}^{k-1-r} \Pi\{\phi^{(n)}(t)\}^{\alpha_n} \right| dt.\right)$$
$$= I_{r,1}(u) + I_{r,2}(u), \text{ say.}$$

We have, by Lemma 3,

$$I_{r,1}(u) = o[\Pi |u^{n} \phi^{(n)}(u)|^{\alpha_n}]$$
$$= o[\Pi\{\phi(u)\}^{\alpha_n}]$$
$$= o[\{\phi(u)\}^{\Sigma\alpha_n}]$$
$$= o[\{\phi(u)\}^{k-1}].$$

Also,

$$I_{r,2}(u)=o\left(\int_{\lambda_0}^{u} t^k\{\phi(u)-\phi(t)\}^{k-1-r-1}\phi'(t)\,\Pi\,|\phi^{(n)}(t)|^{\alpha_n}dt\right)$$

$$+o\left(\int_{\lambda_0}^{u} t^k\{\phi(u)-\phi(t)\}^{k-1-r}\Sigma\left|\frac{\phi^{(n+1)}(t)}{\phi^{(n)}(t)}\right|\alpha_n\Pi\,|\phi^{(n)}(t)|^{\alpha_n}dt\right)$$

$$=o\left(\int_{\lambda_0}^{u}\{\phi(u)-\phi(t)\}^{k-1-r-1}\phi'(t)\Pi\,|t^n\phi^{(n)}(t)|^{\alpha_n}dt\right)$$

$$+o\left(\int_{\lambda_0}^{u}\{\phi(u)-\phi(t)\}^{k-1-r}\,\Sigma\alpha_n\,t^n\,|\phi^{(n+1)}(t)|\,|t^n\,\phi^{(n)}(t)|^{\alpha_n-1}\right.$$

$$\left.\ldots|t^k\phi^{(k)}(t)|\ldots dt\right)$$

Hence, by the hypothesis (a) and Lemma 3, we get

$$I_{r,2}(u)=o\,[\{\phi(u)\}^{\Sigma\alpha_n}]\int_{\lambda_0}^{u}\{\phi(u)-\phi(t)\}^{k-1-r-1}\phi'(t)\,dt$$

$$+o\left(\int_{\lambda_0}^{u}\{\phi(u)-\phi(t)\}^{k-1-r}t^n\,|\phi^{(n+1)}(t)\,|\alpha_n|\,\phi(t)|^{\Sigma\alpha_n-1}dt\right)$$

$$=o\,[\{\phi(u)\}^r]\{\phi(u)\}^{k-1-r}+o\,[\{\phi(u)\}^{k-1-r+(r-1)}]$$

$$\times\int_{\lambda_0}^{u}t^n\,|\phi^{(n+1)}(t)|\,dt$$

$$=o\,[\{\phi(u)\}^{k-1}].$$

Therefore

$$\int^{\omega}\phi'(u)\,|I_r(u)|\,du=o\left(\int^{\omega}\{\phi(u)\}^{k-1}\phi'(u)\,du\right)=o[\{\phi(w)\}^k].$$

Case (ii) : *k is not an integer.* The required condition (4.1.2) can also be written as

$$\int^{\omega}|dA_{\mu}{}^{k}(u)|=o\,(\omega^k),$$

which, by a change of the variables, reduces to

(4.1.8) $$\int^{w}|dA_{\mu}{}^{k}\{\phi(x)\}|=o\,[\{\phi(w)\}^k].$$

Now, by integration by parts, $A_{\mu}{}^{k}\{\phi(x)\}=\int_{\lambda_0}^{x}\{\phi(x)-\phi(t)\}^k dA_{\lambda}(t)$

$$=K\int_{\lambda_0}^{x}A_{\lambda}{}^{h}(t)\left(\frac{d}{dt}\right)^{h+1}\{\phi(x)-\phi(t)\}^k\,dt,$$

where K denotes a constant which may be different at different places. By Lemma 4

$$A_{\mu}{}^{k}\{\phi(x)\}=\Sigma_{r\neq h+1}K\int_{\lambda_0}^{x}A_{\lambda}{}^{h}(t)P_r(x,t)dt+K\int_{\lambda_0}^{x}A_{\lambda}{}^{h}(t)P_{h+1}(x,t)dt$$

$$=I_1(x)+I_2(x).$$

In order to establish (4.1.8) we proceed to prove that

(4.1.9) $$\int^{w}|dI_1(x)|=o\,[\{\phi(w)\}^k]$$

and

(4.1.10) $$\int^{w}|dI_2(x)|=o\,[\{\phi(w)\}^k].$$

Proof of (4.1.9). We have

$$\int^{w}|dI_1(x)|\leqslant \Sigma K\int^{w}dx\left|\int_{\lambda_0}^{x} A_\lambda^h(t)\,\{\phi(x)-\phi(t)\}^{k-r-1}\phi'(x)\Pi\{\phi^{(n)}(t)\}^{\alpha_n}dt\right|$$

$$\leqslant \Sigma K\int^{w}dx\int_{\lambda_0}^{x}|A_\lambda^h(t)|\{\phi(x)-\phi(t)\}^{k-r-1}\phi'(x)\Pi|\phi^{(n)}(t)|^{\alpha_n}dt.$$

Changing the order of integration, we get

$$\int_{\lambda_0}^{w}|dI_1(x)|\leqslant \Sigma K\int_{\lambda_0}^{w}|A_\lambda^h(t)|\Pi|\phi^{(n)}(t)|^{\alpha_n}dt\int_{t}^{w}\{\phi(x)-\phi(t)\}^{k-r-1}\phi'(x)dx$$

$$=\Sigma K\int_{\lambda_0}^{w}|A_\lambda^h(t)|\Pi|\phi^{(n)}(t)|^{\alpha_n}\{\phi(w)-\phi(t)\}^{k-r}dt$$

$$=O\,[\{\phi(w)\}^{k-r}]\int_{\lambda_0}^{w}|A_\lambda^h(t)|\Pi|\phi^{(n)}(t)|^{\alpha_n}dt.$$

By the first theorem of consistency for strong Riesz summability[1] we have, under the hypothesis,

(4.1.11) $$\int^{w}|A_\lambda^h(t)|dt=o(w^{h+1}).$$

Using the above and Lemma 3, we obtain

$$\int_{\lambda_0}^{w}|A_\lambda^h(t)|\Pi|\phi^{(n)}(t)|^{\alpha_n}dt$$

$$=o\,[t^{h+1}\Pi|\phi^{(n)}(t)|^{\alpha_n}]^w+o\left(\int_{\lambda_0}^{w} t^{h+1}\Sigma\alpha_n|\phi^{(n+1)}(t)|\,|\phi^{(n)}(t)|^{\alpha_n-1}\ldots|\phi^{(k)}(t)|^{\alpha_k}\ldots dt\right)$$

$$=o\,[|t^n\phi^{(n)}(t)|^{\alpha_n}]^w+o\left(\int_{\lambda_0}^{w} t^n|\phi^{(n+1)}(t)|\{\phi(t)\}^{\Sigma\alpha_n-1}dt\right)$$

$$=o\,[\{\phi(w)\}^r]+o\,[\{\phi(w)\}^{r-1}]\int_{\lambda_0}^{w} t^n|\phi^{(n+1)}(t)|dt$$

$$=o\,[\{\phi(w)\}^r]+o\,[\{\phi(w)\}^{r-1}]\phi(w)$$

$$=o\,[\{\phi(w)\}^r].$$

Thus (4.1.9) is established.

1. Srivastava (11).

Proof of (4.1.10). *Case* (1): *$\phi'(t)$ is monotonic increasing.* By Lemma 2, we get

$$I_2(x)=K\int_{\lambda_0}^{x} A_\lambda^h(t)P_{h+1}(t)\,dt$$

$$=K\int_{\lambda_0}^{x} A_\lambda^h(t)\{\phi(x)-\phi(t)\}^{k-h-1}\{\phi'(t)\}^{h+1}dt$$

$$=K\int_{\lambda_0}^{x} \{\phi(x)-\phi(t)\}^{k-h-1}\{\phi'(t)\}^{h+1}dt\int^{t} A_\lambda^{k-1}(s)(t-s)^{h-k}ds$$

$$=K\int_{\lambda_0}^{x} A_\lambda^{k-1}(s)\,ds\int_s^x \left\{\frac{\phi(x)-\phi(t)}{x-t}\right\}^{k-h-1}\{\phi'(t)\}^{h+1}(t-s)^{h-k}(x-t)^{k-h-1}dt.$$

Hence it follows from Lemma 5 (i) that

$$\int_{\lambda_0}^{w}|dI_2(x)|\leqslant K\,\overline{\underset{\lambda_0<s}{bd}}\int_s^w \left|d_x\int_s^x\left\{\frac{\phi(x)-\phi(t)}{x-t}\right\}^{k-h-1}\{\phi'(t)\}^{h+1}\right.$$
$$\left.\times(t-s)^{h-k}(x-t)^{k-h-1}dt\right|\int_{\lambda_0}^{w}|A_\lambda^{k-1}(s)|\,ds$$

$$=o\,(w^k)\,\overline{\underset{\lambda_0<s}{bd}}\int_s^w\left|d_x\int_s^x\left\{\frac{\phi(x)-\phi(t)}{x-t}\right\}^{k-h-1}\{\phi'(t)\}^{h+1}(t-s)^{h-k}(x-t)^{k-h-1}dt\right|.$$

Now, putting $t=x\zeta+s(1-\zeta)$ and applying Lemma 5(ii), we obtain

$$\int_s^w\left|d_x\int_s^x\left\{\frac{\phi(x)-\phi(t)}{x-t}\right\}^{k-h-1}\{\phi'(t)\}^{h+1}(t-s)^{h-k}(x-t)^{k-h-1}dt\right|$$

$$=\int_s^w\left|d_x\int_0^1\left\{\frac{\phi(x)-\phi(x\zeta+s\,\overline{1-\zeta})}{(x-s)(1-\zeta)}\right\}^{k-h-1}\{\phi'(x\zeta+s\,\overline{1-\zeta})\}^{h+1}\zeta^{h-k}(1-\zeta)^{k-h-1}d\zeta\right|$$

$$\leqslant\overline{\underset{0<\zeta<1}{bd}}\int_s^w\left|d_x\left\{\frac{\phi(x)-\phi(x\zeta+s\,\overline{1-\zeta})}{(x-s)(1-\zeta)}\right\}^{k-h-1}\{\phi'(x\zeta+s\,\overline{1-\zeta})\}^{h+1}\right|$$
$$\times\int_0^1\zeta^{h-k}(1-\zeta)^{k-h-1}d\zeta.$$

Since $\phi'(t)$ is monotonic increasing,

$$\{\phi'(x\zeta+s\,\overline{1-\zeta})\}^{h+1}$$

is a monotonic increasing function of x and

$$\left\{\frac{\phi(x)-\phi(x\zeta+s\,\overline{1-\zeta})}{(x-s)(1-\zeta)}\right\}^{k-h-1}$$

is a monotonic decreasing function of x by Lemma 6. We write

$$U(x)=\{\phi'(x\zeta+s\,\overline{1-\zeta})\}^{h+1},\quad V(x)=\left\{\frac{\phi(x)-\phi(x\zeta+s\,\overline{1-\zeta})}{(x-s)(1-\zeta)}\right\}^{k-h-1}.$$

By Lemma 7

$$\int_s^w |d_x\{U(x)\,V(x)\}| \leqslant [U(x)\,V(x)]_{x=s} + [U(x)\,V(x)]_{x=w} + 2\int_s^w U'(x)\,V(x)\,dx.$$

But

$$U(x)\,V(x) = \left[\frac{\phi'(x\zeta+s\,\overline{1-\zeta})}{\{\phi(x)-\phi(x\zeta+s\,\overline{1-\zeta})\}/\{(x-s)(1-\zeta)\}}\right]^{h+1} \left\{\frac{\phi(x)-\phi(x\zeta+s\,\overline{1-\zeta})}{(x-s)(1-\zeta)}\right\}^k$$

$$= \left\{\frac{\phi'(x\zeta+s\,\overline{1-\zeta})}{\phi'(\xi)}\right\}^{h+1} \{\phi'(\xi)\}^k,$$

where $x\zeta+s(1-\zeta) \leqslant \xi \leqslant x$. Therefore

$$U(x)\,V(x) = O\,[\{\phi'(x)\}^k].$$

Hence

$$[U(x)\,V(x)]_{x=s \text{ or } x=w} = O\,[\{\phi'(w)\}^k].$$

Also

$$\int_s^w V(x)\,U'(x)\,dx = \int_s^w \left\{\frac{\phi(x)-\phi(x\zeta+s\,\overline{1-\zeta})}{(x-s)(1-\zeta)}\right\}^{k-h-1}$$

$$\times (h+1)\{\phi'(x\zeta+s\,\overline{1-\zeta})\}^h\, d\phi'(x\zeta+s\,\overline{1-\zeta})$$

$$= \int_s^w \left\{\frac{\phi'(x\zeta+s\overline{1-\zeta})}{\phi'(\xi)}\right\}^{h+1-k} \{\phi'(x\zeta+s\overline{1-\zeta})\}^{k-1}(h+1)\,d\phi'(x\zeta+s\overline{1-\zeta})$$

$$= O[\{\phi'(w\zeta+s\overline{1-\zeta})\}^k]$$

$$= O[\{\phi'(w)\}^k].$$

Hence

$$\int^w |\,dI_2(x)\,| = o(w^k)[\{\phi'(w)\}^k] = o[\{\phi(w)\}^k]$$

as required.

Proof of (4.1.10). *Case* (2): *$\phi'(t)$ is monotonic decreasing.* We take c as in Lemma 8 and write

$$I_2(x) = K\int_{\lambda_0}^{x/c} A_\lambda{}^h(t)\{\phi(x)-\phi(t)\}^{k-h-1}\{\phi'(t)\}^{h+1}dt$$

$$+k\int_{x/c}^{x} A_\lambda{}^h(t)\{\phi(x)-\phi(t)\}^{k-h-1}\{\phi'(t)\}^{h+1}\,dt$$

$$= J_1(x)+J_2(x).$$

Now, for almost all values of x,

$$\frac{d}{dx}J_1(x) = KA_\lambda{}^h(x/c)\{\phi(x)-\phi(x/c)\}^{k-h-1}\{\phi'(x/c)\}^{h+1}$$

$$+K\int_{\lambda_0}^{x/c} A_\lambda{}^h\,(t)\{\phi(x)-\phi(t)\}^{k-h-2}(k-h-1)\phi'(x)\{\phi'(t)\}^{h+1}dt.$$

Therefore

$$\int^{w}\left|\frac{d}{dx}J_1(x)\right|dx\leqslant K\int^{w}|A_\lambda{}^h(x/c)|\{\phi(x)-\phi(x/c)\}^{k-h-1}\{\phi'(x/c)\}^{h+1}\,dx$$

$$+K\int^{w}dx\int_{\lambda_0}^{x/c}|A_\lambda{}^h(t)|\{\phi(x)-\phi(t)\}^{k-h-2}\phi'(x)\{\phi'(t)\}^{h+1}\,dt$$

$$=P(w)+Q(w),\ \text{say}.$$

We have

$$P(w)=K\int^{w}|A_\lambda{}^h(x/c)|\left\{\frac{\phi(x)-\phi(x/c)}{x-x/c}\right\}^{k-h-1}x^{k-h-1}\{\phi'(x/c)\}^{h+1}\,dx$$

$$\leqslant K\int^{w}|A_\lambda{}^h(x/c)|\{\phi'(x)\}^{k-h-1}x^{k-h-1}\{\phi'(x/c)\}^{h+1}\,dx$$

$$=K\int^{w/c}|A_\lambda{}^h(v)|\{\phi'(v)\}^{k-h-1}v^{k-h-1}\{\phi'(v)\}^{h+1}\,dv.$$

Hence, by Lemma 8 and (4.1.11),

$$P(w)\leqslant K\int^{w}|A_\lambda{}^h(v)|\{\phi'(v)\}^{k}v^{k-h-1}\,dv$$

$$=o(w^{h+1})w^{k-h-1}\{\phi'(w)\}^{k}+o\left(\int^{w}v^{h+1+k-h-1}\{\phi'(v)\}^{k-1}|\phi''(v)|\,dv\right)$$

$$+o\left(\int^{w}v^{h+1+k-h-2}\{\phi'(v)\}^{k-1}\phi'(v)\,dv\right)$$

$$=o[\{\phi(w)\}^{k}]+o[\{\phi(w)\}^{k-1}\phi(w)]$$

$$=o[\{\phi(w)\}^{k}],$$

since $k>1$. For $Q(w)$, changing the order of integration, we obtain

$$Q(w)\leqslant K\int_0^{w/c}|A_\lambda{}^h(t)|\{\phi'(t)\}^{h+1}\,dt\int_{ct}^{w}\{\phi(x)-\phi(t)\}^{k-h-2}\phi'(x)\,dx$$

$$\leqslant K\int_0^{w/c}|A_\lambda{}^h(t)|\{\phi(t)\}^{h+1}\{\phi(ct)-\phi(t)\}^{k-h-1}\,dt.$$

Hence, from the proof given for $P(w)$ it follows that

$$Q(w)=o[\{\phi(w)\}^{k}].$$

We proceed to consider $J_2(x)$. By Lemma 2,

$$J_2(x)=K\int_{x/c}^{x}A_\lambda{}^h(t)\{\phi(x)-\phi(t)\}^{k-h-1}\{\phi'(t)\}^{h+1}\,dt$$

$$=K\int_{x/c}^{x}\{\phi(x)-\phi(t)\}^{k-h-1}\{\phi'(t)\}^{h+1}\,dt\int_0^{t}A_\lambda{}^{k-1}(s)(t-s)^{h-k}\,ds.$$

Changing the order of integration, we get

$$\mathcal{J}_2(x)=K\int_0^{x/c} A_\lambda^{k-1}(s)\,ds\int_{x/c}^{x}\{\phi(x)-\phi(t)\}^{k-h-1}(t-s)^{h-k}\{\phi'(t)\}^{h+1}dt$$

$$+K\int_{x/c}^{x} A_\lambda^{k-1}(s)\,ds\int_s^x\{\phi(x)-\phi(t)\}^{k-h-1}(t-s)^{h-k}\{\phi'(t)\}^{h+1}dt$$

$$=\mathcal{J}_{2,1}(x)+\mathcal{J}_{2,2}(x),\text{ say.}$$

Now

$$\int^w\left|\frac{d}{dx}\mathcal{J}_{2,1}(x)\right|dx\leqslant K\int^w|A_\lambda^{k-1}(x/c)|\,dx\int_{x/c}^{x}\{\phi(x)-\phi(t)\}^{k-h-1}(t-x/c)^{h-k}\times\{\phi'(t)\}^{h+1}dt.$$

$$+K\int^w dx\int_0^{x/c}|A_\lambda^{k-1}(s)|\,ds\left|\frac{d}{dx}\int_{x/c}^{x}\{\phi(x)-\phi(t)\}^{k-h-1}(t-s)^{h-k}\{\phi'(t)\}^{h+1}dt\right|.$$

Putting $t=x\zeta+(x/c)(1-\zeta)$ in the above, we obtain

$$\int^w\left|\frac{d}{dx}\mathcal{J}_{2,1}(x)\right|dx\leqslant K\int^w|A_\lambda^{k-1}(x/c)|\,dx\int_0^1\left\{\frac{\phi(x)-\phi(x\zeta+\overline{1-\zeta}\,x/c)}{x(1-1/c)(1-\zeta)}\right\}^{k-h-1}\times\{\phi'(x\zeta+\overline{1-\zeta}\,x/c)\}^{h+1}\zeta^{h-k}(1-\zeta)^{k-h-1}d\zeta$$

$$+K\int^w dx\int_0^{x/c}|A_\lambda^{k-1}(s)|\,ds\left|\frac{d}{dx}\int_0^1\left\{\frac{\phi(x)-\phi(x\zeta+\overline{1-\zeta}\,x/c)}{x(1-1/c)(1-\zeta)}\right\}^{k-h-1}\times\{\phi'(x\zeta+\overline{1-\zeta}\,x/c)\}^{h+1}\left\{\frac{x/c-s}{x-x/c}+\zeta\right\}^{h-k}(1-\zeta)^{k-h-1}d\zeta\right|$$

$=M(w)+N(w)$, say.

Then, by Lemma 8,

$$M(w)\leqslant K\int^w|A_\lambda^{k-1}(x/c)|\{\phi'(x/c)\}^k\,dx$$

$$\leqslant K\int^w|A_\lambda^{k-1}(v)|\,\{\phi'(v)\}^k\,dv$$

$$=o[\{\phi(w)\}^k],$$

by the hypotheses and Lemma 3. And

$$N(w)=K\int^w dx\int_0^{x/c}|A_\lambda^{k-1}(s)|\,ds\int_0^1(1-\zeta)^{k-h-1}\left|\frac{d}{dx}\left\{\frac{x/c-s}{x-x/c}+\zeta\right\}^{h-k}\times\left\{\frac{\phi(x)-\phi(x\zeta+\overline{1-\zeta}\,x/c)}{x(1-1/c)(1-\zeta)}\right\}^{k-h-1}\{\phi'(x\zeta+\overline{1-\zeta}\,x/c)\}^{h+1}\right|d\zeta$$

$$=K\int_0^{w/c}|A_\lambda^{k-1}(s)|\,ds\int_0^1(1-\zeta)^{k-h-1}d\zeta\int_{cs}^{w}\left|\frac{d}{dx}F(x)\,U(x)\,V(x)\right|dx,$$

where

$$F(x)=\left(\frac{x/c-s}{x-x/c}+\zeta\right)^{h-k},\quad U(x)=\left\{\frac{\phi(x)-\phi(x\zeta+\overline{1-\zeta}\,x/c)}{x(1-1/c)(1-\zeta)}\right\}^{k-h-1}$$

and

$$V(x)=\{\phi'(x\zeta+\overline{1-\zeta}\,x/c)\}^{h+1}.$$

Now, it is easily seen that $F(x)$ is a monotonic decreasing function of x. Further, since $\phi'(t)$ is monotonic decreasing, $V(x)$ is a monotonic decreasing function of x and $U(x)$ is a monotonic increasing function of x. Hence

$$\int_{cs}^{w}\left|\frac{d}{dx}\{F(x)\,U(x)\,V(x)\}\right|dx\leqslant\int_{cs}^{w}U(x)\,V(x)\,|F'(x)|\,dx$$
$$+\int_{cs}^{w}F(x)\,|\{d\,U(x)\,V(x)\}|$$
$$\leqslant \overline{\operatorname{bd}}_{cs\leqslant x\leqslant w}\,U(x)\,V(x)\,F(cs)+F(cs)\int_{cs}^{w}\left|d\left\{U(x)\,V(x)\right\}\right|.$$

Therefore, by Lemma 7, and then by Lemma 8,

$$\int_{cs}^{w}\left|\frac{d}{dx}\{F(x)\,U(x)\,V(x)\}\right|dx\leqslant KF(cs)\,\overline{\operatorname{bd}}_{cs\leqslant x\leqslant w}\,U(x)\,V(x)+KF(cs)\int_{cs}^{w}U(x)\{-V'(x)\}dx$$
$$\leqslant K\zeta^{h-k}\,\overline{\operatorname{bd}}_{cs\leqslant x\leqslant w}\,\{\phi'(x)\}^{k-h-1}\{\phi'(x/c)\}^{h+1}+K\zeta^{h-k}\int_{cs}^{w}\{\phi'(x)\}^{k-h-1}.$$
$$\times\{\phi'(x\zeta+\overline{1-\zeta}x/c)\}^{h}\{-d\phi'(x\zeta+\overline{1-\zeta}x/c)\}$$
$$\leqslant k\zeta^{h-k}\,\overline{\operatorname{bd}}_{cs\leqslant x\leqslant w}\,\{\phi'(x/c)\}+K\zeta^{h-k}\int_{cs}^{w}\phi'(x\zeta+\overline{1-\zeta}x/c)\}^{k-1}\{-d\phi'(x\zeta+\overline{1-\zeta}x/c)\}$$
$$\leqslant K\zeta^{h-k}\{\phi'(s)\}^{k}.$$

Hence

$$\mathcal{N}(w)\leqslant K\int_{0}^{w/c}|A\lambda^{k-1}(s)|\{\phi'(s)\}^{k}ds\int_{0}^{1}(1-\zeta)^{k-h-1}\zeta^{h-k}d\zeta$$
$$=K\int_{0}^{w/c}|A\lambda^{k-1}(s)|\{\phi'(s)\}^{k}ds$$
$$=o[\{\phi(w)\}^{k}].$$

Now we have only to show that

$$\int^{w}\left|\frac{d}{dx}\mathcal{J}_{2,\,2}(x)\right|dx=o[\{\phi(w)\}^{k}].$$

As before

$$\int^{w}\left|\frac{d}{dx}\mathcal{J}_{2,\,2}(x)\right|dx\leqslant K\int^{w}|A\lambda^{k-1}(x/c)|dx\int_{x/c}^{x}\{\phi(x)-\phi(t)\}^{k-h-1}$$
$$\times(t-x/c)^{h-k}\{\phi'(t)\}^{h+1}dt$$
$$+K\int^{w}dx\int_{x/c}^{x}|A\lambda^{k-1}(s)|\,ds\left|\frac{d}{dx}\left(\int_{s}^{x}\{\phi(x)-\phi(t)\}^{k-h-1}(t-s)^{h-k}\{\phi'(t)\}^{h-1}dt\right)\right|$$
$$=M^{*}(w)+\mathcal{N}^{*}(w),\text{ say.}$$

From the proof given for $M(w)$ it follows that

$$M^*(w)=o[\{\phi(w)\}^k].$$

Further, putting $t=x\zeta+s\overline{1-\zeta}$, we get

$$N^*(w)=K\int^w dx\int_{x/c}^x |A_\lambda^{k-1}(s)|\,ds\left|\frac{d}{dx}\left(\int_0^1\left\{\frac{\phi(x)-\phi(x\zeta+s\overline{1-\zeta})}{(x-s)(1-\zeta)}\right\}^{k-h-1}\right.\right.$$
$$\left.\left.\times\{\phi'(x\zeta+s\overline{1-\zeta})\}^{h+1}(1-\zeta)^{k-h-1}\zeta^{h-k}d\zeta\right)\right|$$

$$=K\int^w dx\int_{x/c}^x |A_\lambda^{k-1}(s)|\,ds\left|\int_0^1 (1-\zeta)^{k-h-1}\zeta^{h-k}\left|\frac{d}{dx}\{U_1(x)V_1(x)\}\right|d\zeta\right|$$

where

$$U_1(x)=\left\{\frac{\phi(x)-\phi(x\zeta+s\overline{1-\zeta})}{(x-s)(1-\zeta)}\right\}^{k-h-1}, \quad V_1(x)=\{\phi'(x\zeta+s\overline{1-\zeta})\}^{h+1}.$$

Changing the order of integration, we get

$$N^*(w)=\int^{w/c}|A_\lambda^{k-1}(s)|\,ds\int_s^{cs}dx\int_0^1\zeta^{h-k}(1-\zeta)^{k-h-1}\left|\frac{d}{dx}\{U_1(x)\;V_1(x)\}\right|d\zeta$$
$$+\int_{w/c}^w|A_\lambda^{k-1}(s)|\,ds\int_s^w dx\int_0^1\zeta^{h-k}(1-\zeta)^{k-h-1}\left|\frac{d}{dx}\{U_1(x)V_1(x)\}\right|d\zeta$$
$$=\int^{w/c}|A_\lambda^{k-1}(s)|\,ds\int_0^1\zeta^{h-k}(1-\zeta)^{k-h-1}d\zeta\int_s^{cs}\left|\frac{d}{dx}\{U_1(x)\;V_1(x)\}\right|dx$$
$$+\int_{w/c}^w|A_\lambda^{k-1}(s)|\,ds\int_0^1\zeta^{h-k}(1-\zeta)^{k-h-1}d\zeta\int_s^w\left|\frac{d}{dx}\{U_1(x)V_1(x)\}\right|dx$$

Now, since $\phi'(t)$ is monotonic decreasing, $V_1(x)$ is monotonic decreasing and $U_1(x)$ is monotonic increasing by Lemma 6. Therefore

$$\int_s^w|d\{U_1(x)V_1(x)\}|\leqslant K\underset{s\leqslant x\leqslant\omega}{\overline{bd}}\;U_1(x).\underset{s\leqslant x\leqslant\omega}{\overline{bd}}\;V_1(x)$$
$$=KU_1(w)V_1(s)$$
$$=K\{\phi'(w)\}^{k-h-1}\{\phi'(s)\}^{h+1}.$$

Hence, using Lemma 8 and the condition (4.1.1), we obtain

$$N^*(w)\leqslant K\int^{w/c}|A_\lambda^{k-1}(s)|\,\{\phi'(cs)\}^{k-h-1}\{\phi'(s)\}^{h+1}ds$$
$$+K\int_{w/c}^w|A_\lambda^{k-1}(s)|\,\{\phi'(w)\}^{k-h-1}\{\phi'(s)\}^{h+1}ds$$

$$\leqslant K\int^{w}|A_\lambda^{k-1}(s)|\,\{\phi'(s)\}^k\,ds+K\int_{w/c}^{w}\{\phi'(w)\}^{k-h-1}\{\phi'(w/c)\}^{h+1}\,|A_\lambda^{k-1}(s)|\,ds$$

$$=o\,[\{\phi(w)\}^k]+o\,[w^k\{\phi'(w)\}^k]\left\{\frac{\phi'(w/c)}{\phi'(w)}\right\}^{h+1}$$

$$=o\,[\{\phi(w)\}^k].$$

This completes the proof of (4.1.10).

4.2. *Proof of Theorem* 2. The series

$$\sum^{\infty}\frac{\cos nx}{(\log n)^{\epsilon}},\ \epsilon>0,\ 0<x<\pi,$$

is convergent, or summable $(C, 0)$. Hence the series

$$\sum^{\infty}\frac{\cos nx}{n^{1-\alpha}(\log n)^{\epsilon}},\ 0<\alpha<1, \tag{4.2.1}$$

is summable $(C, \alpha-1)$[1] and therefore it is summable $[C, \alpha]$[2] or summable $[R, n, \alpha]$ on account of the equivalence of summabilities $[C, \alpha]$ and $[R, n, \alpha]$.[3] We now show that the series (4.2.1) is not summable $[R, \log n, \alpha]$. For if it were so, the series

$$\sum^{\infty}\frac{\cos nx}{n^{1-\alpha}(\log n)^{\epsilon+\epsilon_1+\alpha}},\ \epsilon_1>0$$

would be summable $|R, n, \alpha|$[4] or summable $|C, \alpha|$.[5] We take ϵ and ϵ_1 such that $\alpha+\epsilon+\epsilon_1=1$. Then, by a result of Kogbetliantz,[6]

$$\sum\frac{\cos nx}{n\log n}$$

would be absolutely convergent, which is not true.

1. Hardy (4), Theorem 76.
2. Winn (15).
3. Boyd and Hyslop (1).
4. Tatchell (13).
5. Hyslop (7).
6. Kogbetliantz (8).

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A NOTE ON SYMMETRIC AFFINE CONNECTIONS

By

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1. Consider an n-dimensional Riemannian space having the metric tensor g_{ij} which admits an arbitrary parallel displacement of vectors defined by

$$dV^t+\Gamma_{ij}^t V^i dx^j=0. \tag{1.1}$$

Let the covariant derivative with respect to (1.1) be denoted by a comma and the summation convention with regard to dummy indices be adopted. The following facts are known.[1] If we put

$$a=\Gamma_{ij}^t,\ a^*=\Gamma_{ij}^t+g^{mt}g_{im,j},\ a'=\Gamma_{ji}^t,$$

then for the parallelism (1.1) corresponding to a, there exist uniquely the two parallelisms corresponding to a* and a′ which are respectively called the associate ond conjugate of a having the property

$$a^{**}=a''=a.$$

If particular, a is called self-associate if $a=a^*$ and self-conjugate if $a=a'$. Further, if we put

$$\alpha=g^{mt}g_{im,j},\ \alpha_c=g^{mt}g_{jm,i},\ \gamma=g^{mt}g_{ij,m},$$

$$\beta=g^{mt}g_{is}(\Gamma_{mj}^s-\Gamma_{jm}^s),\ \beta_c=g^{mt}g_{js}(\Gamma_{mi}^s-\Gamma_{im}^s),$$

and construct the sequence

$$a_1=a,\ a_2=a^*,\ a_3=a^{*\prime},\ a_4=a^{*\prime *},\ a_5=a^{*\prime *\prime},\ \ldots .$$

then the sequence is a finite cyclic sequence of 12 terms (assuming all terms distinct) having the following values:

$$a_1=a,\ a_2=a+\alpha,\ a_3=a'+\alpha_c,\ a_4=a+\alpha+\beta-\gamma,$$

$$a_5=a'+\alpha_c+\beta_c-\gamma,\ a_6=a+\alpha+\alpha_c+\beta+\beta_c-\gamma,$$

$$a_7=a'+\alpha+\alpha_c+\beta+\beta_c-\gamma,\ a_8=a'+\alpha_c+\beta+\beta_c-\gamma,$$

$$a_9=a+\alpha+\beta+\beta_c-\gamma,\ a_{10}=a'+\alpha_c+\beta_c,\ a_{11}=a+\alpha+\beta,\ a_{12}=a'.$$

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1. Sen (1).

Finally, if we put $u=\{{}_{i}{}^{t}{}_{j}\}$, the Christoffel symbol, then

(1.2) $$u=\tfrac{1}{2}(a_i+a_{i+6}),\ i=1,2,\ldots,6.$$

Evidently u is both self-associate and self-conjugate.

2. Let us now make the following specialisation. Suppose first that the generating element a is made self-conjugate by imposing the condition $a_1=a_{12}$. This makes $\beta=\beta_c=0$. Then the terms of the above sequence reduce to the following six (assuming all distinct):

(2.1) $$a_1=a_{12}=a,\ a_2=a_{11},\ a_3=a_{10},\quad a_4=a_9,\ a_5=a_8,\ a_6=a_7=b, \text{ say.}$$

Thus a self-conjugate a makes b also self-conjugate as is to be expected from (1.2). Next suppose that the condition $a_2=a_3$ be further imposed so as to make the remaining elements also self-conjugate. This makes $\alpha=\alpha_c=\gamma$. We are, therefore, left with the following two self-conjugate elements:

(2.2) $$a_1=a_4=a_5=a_8=a_9=a_{12}=a\quad a_2=a_3=a_6=a_7=a_{10}=a_{11}=c, \text{ say.}$$

Now let us introduce three orthogonal ennuples ${}^{p}h_{\lambda}|_i$, $\lambda=1, 2, 3$, in the space satisfying the usual equations

$$g_{ij}=\sum_{p}{}^{p}h_{\lambda}|_i{}^{p}h_{\lambda}|_j,\ {}^{p}h_{\lambda}{}^{i}|=g^{ij}\,{}^{p}h_{\lambda}|_j,\ \lambda=1,2,3.$$

Let us take

(2.3) $$a=\tfrac{1}{2}{}^{p}h_1{}^{t}|\left(\frac{\partial\,{}^{p}h_{1}|_i}{\partial x^j}+\frac{\partial\,{}^{p}h_{1}|_j}{\partial x^i}\right)$$

and look for the conditions under which we can take

(2.4) $$b=\tfrac{1}{2}{}^{p}h_2{}^{t}|\left(\frac{\partial\,{}^{p}h_{2}|_i}{\partial x^j}+\frac{\partial\,{}^{p}h_{2}|_j}{\partial x^i}\right),\quad c=\tfrac{1}{2}{}^{p}h_3{}^{t}|\left(\frac{\partial\,{}^{p}h_{3}|_i}{\partial x^j}+\frac{\partial\,{}^{p}h_{3}|_j}{\partial x^i}\right)$$

satisfying the relations (2.1) and (2.2) respectively. We notice initially that if the covariant derivative of g_{ij} with respect to the parallelism corresponding to (2.3) be denoted by a comma, then

$$g_{ij,k}+g_{jk,i}+g_{ki,j}=0.$$

If therefore $g_{ij,k}$ has to be symmetric in all the indices, then $g_{ij,k}=0$ Consequently a becomes both self-conjugate and self-associate. Accordingly, it follows from (1.2), (2.1) and (2.2) that a, b, c as given by (2.3) (2.4) must satify the relations

(2.5) $$u=\tfrac{1}{2}(a+b),\ u=a=c.$$

The conditions (2.5) may be put in another form which has a geometrical interpretation. Let the covariant derivative with respect to Levi-Civita parallelism be denoted by curly brackets. It is then known that Ricci's coefficients of rotation with respect to an orthogonal ennuple ${}^{p}h_i$ are given by

$$\gamma_{pqr}=\{{}^{p}h_l\}_m \, {}_{q}h^l {}_{r}h^m.$$

These coefficients, which are skew in the first two indices, may be regarded as an ennuplet tensor, and one can derive from them a covariant tensor by composition with the h's as

$$(2.6) \qquad \gamma_{ijk}=\gamma_{pqr}\, {}^{p}h_i {}^{q}h_j {}^{r}h_k.$$

In view of the identity

$$ph^t\left(\frac{\partial {}^{p}h_i}{\partial x^j}-\{{}^{p}h_i\}_j\right)=\left\{ {t \atop ij} \right\},$$

the conditions (2.5) may be written as

$$ph_1{}^t|(\{{}^{p}h_{1|i}\}_j+\{{}^{p}h_{1|j}\}_i)+ph_2{}^t|(\{{}^{p}h_{2|i}\}_j+\{{}^{p}h_{2|j}\}_i)=0,$$
$$ph_3{}^t|(\{{}^{p}h_{3|i}\}_j+\{{}^{p}h_{3|j}\}_i)=0.$$

If then $\gamma_{\lambda|pqr}$ denote the coefficients of rotations with respect to the ennuples ${}^{p}h_{\lambda|i}$, $\lambda=1, 2, 3$, the above conditions may, in consequence of (2.6), be finally written as

$$(2.7) \qquad \gamma_{1|ijk}+\gamma_{1|ikj}+\gamma_{2|ijk}+\gamma_{2|ikj}=0, \quad \gamma_{3|ijk}+\gamma_{3|ikj}=0.$$

There exist spaces which admit orthogonal ennuples satisfying the conditions (2.5) or its equivalent (2.7).[1] For example, the first of the two conditions are satisfied in the 3-dimensional space given by

$g_{ii}=1/l^2[(x^1)^2+(x^2)^2+(x^3)^2]$, $g_{ij}=0(i\neq j)$, l=a constant, and the second is satisfied in a 3-dimensional space of constant curvature.

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1. Sen (2).

ON THE GIBBS PHENOMENON FOR THE NÖRLUND MEANS

By

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1. Let $f(x)$ be a function of the class $L(0, 2\pi)$ and let its Fourier series be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

with its partial sum

$$s_n(x)=\tfrac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx).$$

The Fourier series is said to exhibit the Gibbs Phenomenon at the point $x=x_0$, if

$$\varliminf_{\substack{x\to x_0\\ n\to\infty}} s_n(x) < \varliminf_{x\to x_0} f(x) \leqslant \varlimsup_{x\to x_0} f(x) < \varlimsup_{\substack{x\to x_0\\ n\to\infty}} s_n(x).$$

This notion can be generalized by replacing the ordinary partial sum $s_n(x)$ by partial sums depending upon a regular method of summation. Thus if $N_n^p(x)$ denote the n-th Nörlund means of the sequence $\{s_n(x)\}$, where

$$N_n^p(x)=\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k(x)$$

we say that the Nörlund mean $N_n^p(x)$ exhibits the Gibbs Phenomenon at the point $x=x_0$, if

$$\varliminf_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x) < \varliminf_{x\to x_0} f(x) \leqslant \varlimsup_{x\to x_0} f(x) < \varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x).$$

If, on the other hand, the inequality

$$\varliminf_{x\to x_0} f(x) \leqslant \varliminf_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x) \leqslant \varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x) \leqslant \varlimsup_{x\to x_0} f(x)$$

holds, the Gibbs Phenomenon does not occur.

The problem of the occurrence and non-occurrence of the Gibbs Phenomenon has been considered by Cramer[1] for the case of Cesàro summability and by Kuttner[2] for Riesz summability and also for general K-methods of summability.

In this paper we consider cognate problems for the regular Nörlund means with positive generating coefficients. We raise the following two questions :—

(i) Suppose we are given two Nörlund methods of summation (N,p) and (N,q) with positive cofficients. Suppose that the Gibbs Phenomenon does not occur for the (N,p) method. Under what conditions, if any, will the non-persistence of the Gibbs Phenomenon for the (N,p) method imply the same for the (N,q) method ?

(ii) What conditions must be satisfied by a given Nörlund method (N,p) in order that the Gibbs Phenomenon should not occur for any function $f(x)$ at $x=x_0$?

The following theorems which we shall establish in the sequel give an answer to these questions :—

THEOREM 1. *Let (N,p) and (N,q) be two regular Nörlund methods of summation with positive coefficients. Let*

(1.1) (i) $k_n \geqslant 0$ for all n,

(1.2) (ii) $\sum_{r=0}^{n} k_r = o(Q_n)$ as $n \to \infty$,

where $\{k_n\}$ is defined by the relations

(1.3) $$\sum_{n=0}^{\infty} k_n z^n = \frac{\sum_{n=0}^{\infty} q_n z^n}{\sum_{n=0}^{\infty} p_n z^n} = \frac{\sum_{n=0}^{\infty} Q_n z^n}{\sum_{n=0}^{\infty} P_n z^n}.$$

If the sequence $\{s_n(x)\}$ is bounded in the neighbourhood of $x=x_0$, then the non-persistence of the Gibbs Phenomenon for the (N,p) method would necessarily imply the same for the (N,q) method.

THEOREM 2. *If $p_n > 0$ and monotonic decreasing, the (N,p) means of the partial sum of the Fourier series of any function $f(x)$ must fail to exhibit the Gibbs Phenomenon.*

1. Cramer (1)
2. Kuttner (2), (3), (4).

1.2. *Proof of Theorem* 1.

If

$$\varliminf_{\substack{x\to x_0\\ n\to\infty}} \mathcal{N}_n^p(x) \leqslant \varliminf_{\substack{x\to x_0\\ n\to\infty}} \mathcal{N}_n^q(x) \leqslant \varlimsup_{\substack{x\to x_0\\ n\to\infty}} \mathcal{N}_n^q(x) \leqslant \varlimsup_{\substack{x\to x_0\\ n\to\infty}} \mathcal{N}_n^p(x),$$

then the non-occurrence of the Gibbs Phenomenon for the $(\mathcal{N}, p)$ method will imply the same for the $(\mathcal{N}, q)$ method.

In order to prove the theorem we have to show that, under the conditions stated, the above inequality holds.

Since

$$\mathcal{N}_n^p(x) = \frac{1}{\mathrm{P}_n} \sum_{\nu=0}^{n} p_{n-\nu}\, s_\nu(x) = \frac{(p, s)_n}{\mathrm{P}_n},$$

$$\mathcal{N}_n^q(x) = \frac{1}{Q_n} \sum_{\nu=0}^{n} q_{n-\nu}\, s_\nu(x) = \frac{(q, s)_n}{Q_n},$$

and (1.3) holds, we have

$$\text{(i)} \qquad \sum_{n=0}^{\infty} k_n z^n \sum_{n=0}^{\infty} P_n z^n = \sum_{n=0}^{\infty} Q_n z^n,$$

whence we deduce that

$$\sum_{\nu=0}^{n} k_{n-\nu} P_\nu = Q_n; \tag{2.1}$$

and

$$\text{(ii)} \quad \sum_{n=0}^{\infty} Q_n \mathcal{N}_n^q(x)\, z^n = \sum_{n=0}^{\infty} (q, s)_n\, z^n$$

$$= \sum_{n=0}^{\infty} q_n z^n \sum_{n=0}^{\infty} s_n(x)\, z^n$$

$$= \sum_{n=0}^{\infty} k_n z^n \sum_{n=0}^{\infty} p_n z^n \sum_{n=0}^{\infty} s_n(x)\, z^n$$

$$= \sum_{n=0}^{\infty} k_n z^n \sum_{n=0}^{\infty} (p, s)_n\, z^n$$

$$=\sum_{n=0}^{\infty} k_n z^n \sum_{n=0}^{\infty} P_n N_n^p(x) z^n$$

$$=\sum_{n=0}^{\infty} (k, PN^p(x))_n z^n,$$

whence we deduce that

$$N_n^q(x)=\frac{1}{Q_n}\sum_{\nu=0}^{n} k_{n-\nu} P_\nu N_\nu^p(x). \tag{2.2}$$

Since the sequence $\{s_n(x)\}$ is bounded in the neighbourhood of the point $x=x_0$, the Nörlund transforms $N_n^p(x)$ and $N_n^q(x)$ are also bounded in the same neighbourhood. Hence the four functional limits

$$\varliminf_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x),\ \varliminf_{\substack{x\to x_0\\ n\to\infty}} N_n^q(x),\ \varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^q(x),\ \varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x)$$

are finite.

First let us prove that

$$\varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^q(x) \leqslant \varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x). \tag{2.3}$$

Let us assume that

$$\varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x)=0,$$

for, if

$$\varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x)=K\neq 0,$$

we can consider the sequence

$$\{N_n^p(x)-K\}.$$

Then our assumption is that the given sequence $\{s_n(x)\}$ is such that

$$\varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x)=0, \tag{2.4}$$

and we wish to prove that

$$\overline{\lim_{\substack{x\to x_0\\ n\to\infty}}}\ N_n^q(x)\leqslant 0.$$

Since (2.4) holds, given an arbitrary $\epsilon>0$, there exists an index $n_0=n_0(\epsilon)$ and $\eta=\eta(\epsilon)$, such that

$$N_n^p(x)<\epsilon,$$

for all $n>n_0$ and all x such that $|x-x_0|<\eta$. Then from (2.2), we have

$$N_n^q(x)=\frac{1}{Q_n}\sum_{\nu=0}^{n} k_{n-\nu}P_\nu N_\nu^p(x)$$

$$=\frac{1}{Q_n}\sum_{\nu=0}^{n_0} k_{n-\nu}P_\nu N_\nu^p(x)+\frac{1}{Q_n}\sum_{\nu=n_0+1}^{n} k_{n-\nu}P_\nu N_\nu^p(x).$$

But

$$\frac{1}{Q_n}\sum_{\nu=n_0+1}^{n} k_{n-\nu}P_\nu N_\nu^p(x)<\frac{\epsilon}{Q_n}\sum_{\nu=n_0+1}^{n} k_{n-\nu}P_\nu$$

$$<\frac{\epsilon}{Q_n}\sum_{\nu=0}^{n} k_{n-\nu}P_\nu \tag{2.5}$$

$$=\epsilon,$$

by virtue of the relation (2.2).

Let us fix n_0 and η so that the above inequality holds. There remains to be considered the sum

$$\frac{1}{Q_n}\sum_{\nu=0}^{n_0} k_{n-\nu}P_\nu N_\nu^p(x).$$

Since n_0 and η are fixed, there exists a constant A such that, for $\nu=0,1\ldots,n$ and $|x-x_0|<\eta$,

$$|P_\nu N_\nu^p(x)|<A.$$

Hence

$$\left|\frac{1}{Q_n}\sum_{\nu=0}^{n_0} k_{n-\nu}P_\nu N_\nu^p(x)\right|\leqslant A\frac{1}{Q_n}\sum_{\nu=0}^{n_0} k_{n-\nu}$$

$$<A\frac{1}{Q_n}\sum_{\nu=0}^{n} k_{n-\nu}. \tag{2.6}$$

Now, since (1.2) holds, combining (2.5) and (2.6), we get

$$\varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^q(x)\leqslant\epsilon.$$

The left hand side of this inequality is independent of ϵ. Consequently

$$\varlimsup_{\substack{x\to x_0\\ n\to\infty}} N_n^q(x)\leqslant 0.$$

The other inequality, viz.,

$$\varliminf_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x)\leqslant \varliminf_{\substack{x\to x_0\\ n\to\infty}} N_n^q(x),$$

can be proved in a similar way by considering

$$\varliminf_{\substack{x\to x_0\\ n\to\infty}} N_n^p(x)=-\varlimsup_{\substack{x\to x_0\\ n\to\infty}} [-N_n^p(x)].$$

This completes the proof of the theorem.

If we choose $p_n=\binom{n+r-1}{n}$ and $q_n=\binom{n+r+s-1}{n}$ with $r>0, s>0$, then the Nörlund methods (N, p) and (N, q) reduce to the Cesàro methods (C, r), $r>0$ and $(C, r+s)$, $r>0, s>0$, respectively. It can be seen that the conditions of the theorem are satisfied. It, therefore, follows that if the Gibbs Phenomenon does not persist for the (C, r) method for $r>0$, it cannot persist for the $(C, r+s)$ method where $s>0$, which is a result obtained by Cramer.[1] Since summability $(C, 1)$ does not exhibit the Gibbs Phenomenon,[2] the preceding result shows that the same is true for summability $(C, 1+s)$, $s>0$.

3. *Proof of Theorem* 2.

If $N_n(t)$ denotes the kernel of the (N, p) mean of the partial sum of the Fourier series, we have

$$(3.1)\qquad N_n(t)=\frac{1}{2\pi P_n}\sum_{k=0}^{n} p_{n-k}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{1}{2}t}.$$

1. See, for instance, Kuttner (3).

2. Cramer (1).

It is known[1] that if the kernel corresponding to any method of summability is positive, then the Gibbs Phenomenon cannot occur whatever *the function $f(x)$ may be.*

Applying Abel's *transformation to the kernel* (3.1), *we get*

$$\mathcal{N}_n(t)=\frac{1}{2\pi P_n}\left[\sum_{k=0}^{n-1}(p_{n-k}-p_{n-k-1})\left(\frac{\sin(k+1)t/2}{\sin\frac{1}{2}t}\right)^2+p_0\left(\frac{\sin(n+1)t/2}{\sin\frac{1}{2}t}\right)^2\right].$$

Since $p_n>0$ and monotonic non-decreasing,

$$\mathcal{N}_n(t)>0$$

for all n and t. This completes the proof of Theorem 2.

If we choose $p_n=\binom{n+r}{r}, r>0$, the corresponding $(\mathcal{N}, p)$ mean reduces to the Cesàro *mean* $(C, 1+r)$, $r>0$. *Since this sequence is positive and* non-decreasing, it follows from this theorem that the Gibbs Phenomenon cannot occur for the Cesàro summability of order $\geqslant 1$.[2]

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1. See, for instance, Hardy and Rogosinski (5); also Kuttner (4).
2. Kuttner loc. cit.

A TAUBERIAN THEOREM FOR ABSOLUTE RIESZ SUMMABILITY

By

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1.1. Let Σa_n be a given infinite series, and let λ_n be a positive, steadily increasing, monotonic function of n, tending to infinity with n. We write

$$A_\lambda(\omega)=A_\lambda{}^0(\omega)=\sum_{\lambda_n\leqslant\omega} a_n,$$

and

$$A_\lambda{}^k(\omega)=\sum_{\lambda_n\leqslant\omega}(\omega-\lambda_n)^k a_n \quad (k>0).$$

The series Σa_n is said to be absolutely summable (R, λ, k), or summable $|R, \lambda, k|$, $k\geqslant 0$, if $A_\lambda{}^k(\omega)/\omega^k$ is of bounded variation in (A, ∞), where A is a finite positive number.[1]

1.2. To Obrechkoff[2] is due the "first theorem of consistency" for absolute Riesz summability, which states that if an infinite series is summable $|R, \lambda, k|$, $k\geqslant 0$, then it is also summable $|R, \lambda, k'|$ for every $k'>k$. However, since it is known that a series which is summable $|R, \lambda, r|$ is not necessarily summable $|R, \lambda, r'|$, where $r'<r$, it would be interesting to obtain "Tauberian conditions" on the general term of Σa_n, under which summability $|R, \lambda, k|$, $k>0$, may imply summability $|R, \lambda, 0|$, that is, the absolute convergence of the series. In this direction Pati[3] has proved the following theorem, which is the analogue for absolute summability of a well-known Tauberian theorem of Hardy.[4]

If (i) Σa_n *is summable* $|R, \lambda, k|$, $k>0$, (ii) $\{a_n\lambda_n/(\lambda_n-\lambda_{n-1})\}$ *is a sequence of bounded variation and* (iii) $\{\lambda_n/\lambda_{n+1}\}$ *is a sequence of bounded variation, then* Σa_n is *absolutely convergent.*

The particular case of the above for $k=1$ was proved by Mohanty.[5]

The object of this paper is to generalise the above result by replacing the condition concerning the bounded variation of the sequence

1. Obrechkoff (5), (6).
2. Obrechkoff (5), (6).
3. Pati (7).
4. Hardy (1).
5. Mohanty (4).

$\{a_n \lambda_n/(\lambda_n - \lambda_{n-1})\}$ by the bounded variation of the sequence $\{\lambda_m^{-1}\Sigma_1^m \lambda_n a_n\}$. It can be seen from Lemma 1 (stated below) that whenever $\{a_n\lambda_n/(\lambda_n - \lambda_{n-1})\}$ is of bounded variation, the sequence $\{\lambda_m^{-1}\Sigma_1^m\lambda_n a_n\}$ is necessarily of bounded variation, but it is easy to see that the converse is not true.

It may be remarked that the theorem of this paper includes the following result[1] as a particular case, since summability $|R, n, k|$ is equivalent to summability $|C, k|$ for $k \geqslant 0$.[2]

If Σa_n is summable $|C|$ and if $\Sigma\Delta(na_n)$ is summable $|C, 1|$, then Σa_n is absolutely convergent.

I take this opportunity of expressing my warmest thanks to Prof. B. N. Prasad for his valuable suggestions during the preparation of this paper.

2.1. We prove the following theorem.

THEOREM. *If* (i) *Σa_n is summable $|R, \lambda, k|$, $k>0$,* (ii) *$\{\lambda_m^{-1}\Sigma_1^m\lambda_n a_n\}$ is a sequence of bounded variation and* (iii) *$\{\lambda_n/\lambda_{n+1}\}$ is a sequence of bounded variation, then Σa_n is absolutely convergent.*

2.2. For proving the theorem we shall require the following lemmas.

LEMMA 1.[3] *Let*

$$\mu_m > 0,$$

$$\lambda_m = \sum_1^m \mu_n,$$

and

$$d_m = \frac{1}{\lambda_m} \sum_1^m \mu_n c_n.$$

Then, if the sequence $\{c_m\}$ is of bounded variation, the sequence $\{d_m\}$ is of bounded variation.

LEMMA 2. *If the sequence $\{\lambda_m/\lambda_{m+1}\}$ is of bounded variation, then the sequence*

$$\frac{1}{\lambda_{m+1}^{s+1}} \sum_1^m \mu_n \lambda_n^s \qquad (s=0, 1, 2, \ldots)$$

is of bounded variation.

The result of this lemma has been proved by Pati.[4]

LEMMA 3. *If* (i) *$\{\lambda_m^{-1}\Sigma_1^m\lambda_n a_n\}$ is a sequence of bounded variation* (ii) *$\{\lambda_m/\lambda_{m+1}\}$ is a sequence of bounded variation, then the sequence*

$$\{\lambda_{m+1}^{-\kappa-1} \sum_{n=1}^m (\lambda_{m+1} - \lambda_n)^\kappa a_n \lambda_n\} \qquad (\kappa=0, 1, 2, \ldots\ldots)$$

1. Hyslop (3).
2. Hyslop (2).
3. Mohanty (4).
4. Pati (7), Proof of Lemma 2.

is of bounded variation.

Proof. Expanding $(\lambda_{m+1}-\lambda_n)^\kappa$, we observe that the lemma will be established if it is proved that the sequence

$$\left\{\frac{1}{\lambda_{m+1}^{s+1}}\sum_{n=1}^{m}\lambda_n^{s+1}a_n\right\}\qquad (s=0, 1, 2, \ldots\ldots\kappa)$$

is of bounded variation. Now

$$\frac{\sum_1^m \lambda_n^{s+1}a_n}{\lambda_{m+1}^{s+1}}=\frac{\sum_{n=1}^{m-1}\Delta\lambda_n^s\sum_{\nu=1}^{n}\lambda_\nu a_\nu+\lambda_m^s\sum_{\nu=1}^{m}\lambda_\nu a_\nu}{\lambda_{m+1}^{s+1}}$$

$$=\frac{\sum_{n=1}^{m-1}\lambda_n\Delta\lambda_n^s(\lambda_n^{-1}\sum_{\nu=1}^{n}\lambda_\nu a_\nu)}{\lambda_{m+1}^{s+1}}+\left(\frac{\lambda_m}{\lambda_{m+1}}\right)^{s+1}\frac{1}{\lambda_m}\sum_{\nu=1}^{m}\lambda_\nu a_\nu$$

$$=P_m+Q_m, \text{ say.}$$

By hypothesis, $\{Q_m\}$ is of bounded variation. Hence, it is sufficient to show that $\{P_m\}$ is of bounded variation. We have

$$P_m=\frac{\sum_{n=1}^{m-1}\lambda_n\Delta\lambda_n^s(\lambda_n^{-1}\sum_{\nu=1}^{n}\lambda_\nu a_\nu)}{\sum_{n=1}^{m-1}\lambda_n\Delta\lambda_n^s}\,\frac{\sum_{n=1}^{m-1}\lambda_n\Delta\lambda_n^s}{\lambda_{m+1}^{s+1}}.$$

Now, by virtue of hypothesis (i) of the lemma and Lemma 1, $\{P_m\}$ is of bounded variation if $\{P_m{}^*\}$ is of bounded variation, where

$$P_m{}^*=\frac{1}{\lambda_{m+1}^{s+1}}\sum_{n=1}^{m-1}\lambda_n\Delta\lambda_n^s.$$

Without loss of generality, we can suppose that $\lambda_1=0$. Now

$$P_m{}^*=\frac{1}{\lambda_{m+1}^{s+1}}\left(\sum_{n=1}^{m}\lambda_n\Delta\lambda_n^s-\lambda_m\Delta\lambda_m^s\right)$$

$$=\frac{1}{\lambda_{m+1}^{s+1}}\left(\sum_{n=1}^{m-1}\mu_{n+1}\lambda_{n+1}^s-\lambda_m\lambda_{m+1}^s-\lambda_m\,\Delta\lambda_m^s\right)$$

$$=\frac{1}{\lambda_{m+1}^{s+1}}\sum_{n=2}^{m}\mu_n\lambda_n^s-\frac{\lambda_m}{\lambda_{m+1}}.$$

Hence, by Lemma 2 and hypothesis (ii) of the lemma, $\{P_m^*\}$ is of bounded variation. This completes the proof of the lemma.

2.3. *Proof of the Theorem.* By virtue of the 'first theorem of consistency' we can assume without any loss of generality that k is a positive integer $\geqslant 1$. Let $k=\kappa+1$, where $\kappa=0, 1, 2, \ldots$.

We have

$$\frac{A_\lambda^{\kappa+1}(\lambda_{m+1})}{\lambda_{m+1}^{\kappa+1}}=\frac{1}{\lambda_{m+1}^{\kappa}}\sum_{n=1}^{m}(\lambda_{m+1}-\lambda_n)^{\kappa}a_n$$

$$-\frac{1}{\lambda_{m+1}^{\kappa+1}}\sum_{n=1}^{m}(\lambda_{m+1}-\lambda_n)^{\kappa}\lambda_n a_n$$

$$=s_m-\sum_{s=0}^{\kappa}\frac{1}{\lambda_{m+1}^{s+1}}\sum_{n=1}^{m}(\lambda_{m+1}-\lambda_n)^{s}\lambda_n a_n.$$

Hence

$$\Sigma|\Delta s_m|\leqslant\sum_{s=0}^{\kappa}\Sigma\left|\Delta\left\{\frac{1}{\lambda_{m+1}^{s+1}}\sum_{n=1}^{m}(\lambda_{m+1}-\lambda_n)^{s}\lambda_n a_n\right\}\right|$$

$$+\int_{\lambda_1}^{\infty}\left|\frac{d}{dw}\left\{\frac{A_\lambda^{\kappa+1}(w)}{w^{\kappa+1}}\right\}\right|dw.$$

Thus, by virtue of Lemma 3 and hypothesis (i) of the theorem, the series Σa_n is absolutely convergent. This completes the proof of the theorem.

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ON THE ABSOLUTE RIESZ SUMMABILITY FACTORS OF INFINITE SERIES (I)

By

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1.1. *Definitions.* Let $\{\lambda_n\}$ be a positive, steadily increasing sequence tending to infinity with n, and Σa_n a given series. We write

$$A_\lambda(\omega)=A_\lambda{}^0(\omega)=\sum_{\lambda_n<\omega} a_n,$$

$$A_\lambda(\omega)=0, \qquad \text{for } \omega\leqslant\lambda_1$$

and

$$A_\lambda{}^k(\omega)=\sum_{\lambda_n<\omega}(\omega-\lambda_n)^k a_n$$

$$=\int_{\lambda_1}^{\omega}(\omega-\tau)^k dA_\lambda(\tau),\ k>0.$$

The series Σa_n is said to be summable (R, λ, k), $k\geqslant 0$, to the sum S, if $\lim_{\omega\to\infty} \omega^{-k}A_\lambda{}^k(\omega)=S$, where S is a finite number. If in addition, $\omega^{-k}A_\lambda{}^k(\omega)$ is also of bounded variation in (A, ∞), where A is a finite positive number, the series Σa_n is said to be summable $|R, \lambda, k|$ to the sum S.

In what follows we take functions $\phi(t)$ and $\psi(t)$ to be defined in $(0, \infty)$ and to be as many times differentiable as required, and $\phi(t)$, a non-negative, monotonic increasing function of t, tending to infinity with t.

1.2. *Introduction.* The first important result on the second theorem of consistency was given by Hardy[1] in 1916 in the following form.

Theorem A. If the series Σa_n is summable (R, λ, k), $k\geqslant 0$, to sum S, and μ is a logarithmico-exponential functon of λ such that

$$\mu=O(\lambda^\Delta),$$

where Δ is some constant, then the series Σa_n is summable (R, μ, k) to the same sum S.

In 1932, Hirst[2] generalised Hardy's theorem by replacing μ by a more general function $\phi(t)$. Analogous theorems for the case of absolute

1. Hardy (5).
2. Hirst (7).

summability have been given by Chandrasekharan,[1] Pati[2] and Guha,[3] in the case of integral order of summability, and by Guha[3] and Prasad and Pati[4] in the case of non-integral order. The most general result on the 'second theorem of consistency' for absolutely summable series, known so far, in the case of integral order of summability is the following theorem of Guha.[5]

THEOREM B. If the series Σa_n is summable $|R\ \lambda, k|$, where k is a positive integer and

$$t^k\phi^{(k)}(t)=O\{\phi(t)\},$$

as $t\to\infty$, then the series Σa_n is summable $|R, \phi(\lambda), k|$.

There arises now a question of determining summability factors $\{\epsilon_n\}$ such that the series $\Sigma\, a_n\, \epsilon_n$ is summable $|R, \phi(\lambda), \kappa|$ if the seres Σa_n is summable $|R, \lambda, \kappa|$, when conditions less stringent than the sufficient conditions in Guha's theorem (Theorem B) are satisfied. By way of answering this question, Guha[6] has recently obtained the following theorem.

THEOREM C. If k is a positive integer and $\phi(t)=\mu\{\lambda(t)\}$, where $\mu(t)$ is a logarithmico-exponential function, such that for sufficiently large x and a positive constant Δ

$$\frac{1}{x}<\Delta\frac{\mu'(x)}{\mu(x)},$$

then the $|R, \lambda, k|$ summability of the series $\Sigma\, a_n$ implies the $|R, \phi(\lambda), k|$ summability of the series $\Sigma a_n\, \psi(\lambda_n)$, where

$$\psi(t)=\frac{\phi(t)}{\omega^k\phi'(t)}\,.$$

The object of this paper is to establish a summability-factor theorem for absolute Riesz summability, which includes as a particular case the Theorem C. It may be remarked that the theoiem of this paper is the analogue, for absolute summability, of a recent result, due to Borwein,[7] concerning ordinary Riesz summability of series.

It may also be mentioned that in the special case when $\psi(\omega)=1$, $\gamma(\omega)=\omega$, our theorem reduces to the 'second theorem of consistency' of Guha

1. Chandrasekharan (2).
2. Pati (8).
3. Guha (3).
4. Prasad and Pati (9).
5. Guha (3).
6. Guha (4).
7. Borwein (1).

(Theorem B), while for the case when $\phi(\omega)=e^{\omega}$, $\psi(\omega)=\omega^{-k}$, $\gamma(\omega)=1$, our theorem becomes the same as the following result due to Tatchell.[1]

Theorem D. If $k\geqslant 0$ and the series Σa_n is summable $|R, \lambda, k|$ then the series $\Sigma a_n \lambda_n^{-k}$ is summable $|R, e^{\lambda}, k|$.

I should like to express my indebtedness to Prof. B. N. Prasad for his kind encouragement and valuable guidance during the preparation of this paper.

2.1. We establish the following theorem.

THEOREM. *Let k be a positive integer, and $\phi(\omega)$ and $\psi(\omega)$ be the k-th indefinite integrals for $\omega\geqslant 0$. If the positive function $\gamma(\omega)$ be such that*

(i) $\gamma(\omega)/\omega \in BV(h, \infty)$,

(ii) $\omega^n \psi^{(n)}(\omega)=O\left[\left\{\dfrac{\gamma(\omega)}{\omega}\right\}^{k-n}\right]$, $n=0, 1, 2, \ldots, k, \omega\geqslant h$,

and

(iii) $\{\gamma(\omega)\}^n \phi^{(n)}(\omega)=O[\phi(\omega)]$, $n=1, 2, \ldots, k, \omega\geqslant h$,

where h is some finite positive number, and if the series Σa_n is summable $|R, \lambda, k|$, then the series $\Sigma a_n \psi(\lambda_n)$ will be summable $|R, \phi(\lambda), k|$.

It is evident that the truth or otherwise of the theorem depends only on the behaviour of $\phi(\omega)$ for sufficiently large ω. We may, therefore, alter $\phi(\omega)$, in any finite interval, and there is thus no loss of generality in supposing that $\phi(\lambda_1)=h=\lambda_1$ and $\phi(\omega)$ is the k-th indefinite integral for $\omega\geqslant 0$. We shall, therefore, make use of this assumption throughout this paper.

2.2. To prove the theorem, we require the following lemmas.

LEMMA 1.[2] *If ρ is a positive integer, then*

$$A_{\lambda}(\tau)=\frac{1}{\rho!}\left(\frac{d}{d\tau}\right)^{\rho} A_{\lambda}^{\rho}(\tau).$$

LEMMA 2.[3] *With the assumption that the expressions used have meaning, if*

$$G(x)=\int_h^x \phi(x, u) g(u)\, du,$$

then

$$\int_h^{\infty} |dG(x)| \leqslant \overline{\underset{h<u}{\text{bound}}} \left\{ \phi(u, u)+\int_u^{\infty} |d_x \phi(x, u)| \right\} \int_h^{\infty} |g(u)|\, du.$$

1. Tatchell (10).
2. Hardy and Riesz (6).
3. Tatchell (10).

LEMMA 3. *If n is a positive number and $m \neq 0$, then the n-th derivative of $\{f(x)\}^m$ is a sum of constant multiples of a finite number of terms of the form*

$$\{f(x)\}^{m-r} \prod_{p=1}^{n} \{f(x)^{(p)}\}^{a_p}$$

where $1 \leqslant r \leqslant n$, and the a's are zeros or integers such that

$$\sum_1^p a_p = r \quad \text{and} \quad \sum_1^n p a_p = n.$$

If m is a positive integer, $1 \leqslant r \leqslant \min(m, n)$.

This is a particular case of a result due to Faa di Bruno[1] on the derivative of a function of a function.

3.1. *Proof of the theorem.*

We are given that

$$t^{-k} A_\lambda^k(t) = \int_\lambda^t \left(1 - \frac{u}{t}\right)^k dA_\lambda(u) \in BV(\lambda_1, \infty),$$

and we have to show that

$$t^{-k} B_\phi^k(t) = t^{-k} \sum_{\phi(\lambda_n) < t} \{t - \phi(\lambda_n)\}^k \psi(\lambda_n)\, a_n$$

$$\in BV(\lambda_1, \infty);$$

or, what is the same thing, that

$$\{\phi(t)\}^{-k} B_\phi^k\{\phi(t)\} = \{\phi(t)\}^{-k} \sum_{\phi(\lambda_n) < \phi(t)} \{\phi(t) - \phi(\lambda_n)\}^k \psi(\lambda_n) a_n$$

$$= \{\phi(t)\}^{-k} \int_{\lambda_1}^t \{\phi(t) - \phi(\omega)\}^k \psi(\omega)\, dA_\lambda(\omega)$$

$$\in BV(h, \infty).$$

Now, by Lemma 1,

$$B_\phi^k\{\phi(t)\} = -\int_{\lambda_1}^t \frac{\partial}{\partial\omega} [\{\phi(t) - \phi(\omega)\}^k \psi(\omega)]\, A_\lambda(\omega)\, d\omega$$

$$= -\frac{1}{k!} \int_{\lambda_1}^t \frac{\partial}{\partial\omega} [\{\phi(t) - \phi(\omega)\}^k \psi(\omega)] \left(\frac{d}{d\omega}\right)^k A_\lambda^k(\omega)\, d\omega.$$

Since $A_\lambda^k(\omega)$ and its first $(k-1)$ derivatives vanish at $\omega = \lambda_1$, integrating $(k-1)$-times the last integral, we get

$$B_\phi^k\{\phi(t)\} = \frac{(-1)^k}{k!} \int_{\lambda_1}^t \left(\frac{\partial}{\partial\omega}\right)^k [\{\phi(t) - \phi(\omega)\}^k \psi(\omega)] \frac{d}{d\omega} A_\lambda^k(\omega)\, d\omega$$

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1. de la Vallée Poussin (11), p. 89.

$$=\frac{(-1)^k}{k!}\int_{\lambda_1}^{t}\left(\frac{\partial}{\partial\omega}\right)^k[\{\phi(t)-\phi(\omega)\}^k\psi(\omega)]\frac{d}{d\omega}\left(\frac{\omega^k A_\lambda{}^k(\omega)}{\omega^k}\right)d\omega$$

$$=\frac{(-1)^k}{k!}\int_{\lambda_1}^{t}\left(\frac{\partial}{\partial\omega}\right)^k[\{\phi(t)-\phi(\omega)\}^k\psi(\omega)]\omega^k\frac{d}{d\omega}\left(\frac{A_\lambda{}^k(\omega)}{\omega^k}\right)d\omega$$

$$+\frac{(-1)^k}{(k-1)!}\int_{\lambda_1}^{t}\left(\frac{\partial}{\partial\omega}\right)^k[\{\phi(t)-\phi(\omega)\}^k\psi(\omega)]\omega^{k-1}.\frac{A_\lambda{}^k(\omega)}{\omega^k}d\omega$$

$$=I(t)+\mathcal{J}(t), \text{ say.}$$

Hence to prove the theorem, it is enough to show that, under the hypotheses of the theorem,

$$I(t)/\{\phi(t)\}^k \quad \epsilon \quad BV(h, \infty)$$

and

$$\mathcal{J}(t)/\{\phi(t)\}^k \quad \epsilon \quad BV(h, \infty).$$

Now by Lemma 2,

$$\{\phi(t)\}^{-k}I(t) \quad \epsilon \quad BV(h, \infty),$$

if

$$[\omega^k(\partial/\partial\omega)^k\{[\phi(t)-\phi(\omega)]^k\psi(\omega)\}\{\phi(t)\}^{-k}]_{t=\omega}$$

$$+\int_{\omega}^{\infty}\left|d_t\left[\omega^k\{\phi(t)\}^{-k}\left(\frac{\partial}{\partial\omega}\right)^k\left(\{\phi(t)-\phi(\omega)\}^k\psi(\omega)\right)\right]\right|<K.^{1}$$

But by Lemma 3 and hypotheses, at $t=\omega$,

$$\{\phi(t)\}^{-k}\omega^k\left(\frac{\partial}{\partial\omega}\right)^k\{[\phi(t)-\phi(\omega)]^k\psi(\omega)\}$$

$$=O[\omega^k\psi(\omega)\ \{\phi'(\omega)\}^k\{\phi(\omega)\}^{-k}]$$

$$=O\left[\frac{\omega^k}{\{\gamma(\omega)\}^k}\psi(\omega)\ \{\gamma(\omega)\ \phi'(\omega)\}^k\{\phi(\omega)\}^{-k}\right]$$

$$=O(1).$$

So

$$\{\phi(t)\}^{-k}I(t) \quad \epsilon \quad BV(h, \infty)$$

if

$$\int_{\omega}^{\infty}\left|d_t\left[\omega^k\{\phi(t)\}^{-k}\left(\frac{\partial}{\partial\omega}\right)^k\{[\phi(t)-\phi(\omega)]^k\psi(\omega)\}\right]\right|<K. \tag{3.11}$$

Also, if we write

$$\theta(t,\omega)=\int_{\omega}^{t}\sigma^{k-1}\left(\frac{\partial}{\partial\sigma}\right)^k\left[\{\phi(t)-\phi(\sigma)\}^k\psi(\sigma)\right]d\sigma,$$

1. Throughout the paper, K denotes an absolute constant, not necessarily the same at each occurrence.

$$\int_{\lambda_1}^{t}\left(\frac{\partial}{\partial\omega}\right)^{k}[\{\phi(t)-\phi(\omega)\}^{k}\psi(\omega)]\,\omega^{k-1}\left(\frac{A_\lambda{}^{k}(\omega)}{\omega^{k}}\right)d\omega$$

$$=-\int_{\lambda_1}^{t}\left(\frac{\partial}{\partial\omega}\right)\theta(t,\omega)\{\omega^{-k}A_\lambda{}^{k}(\omega)\}\,d\omega$$

$$=-\Big[\theta(t,\omega)\,\omega^{-k}A_\lambda{}^{k}(\omega)\Big]_{\lambda_1}^{t}+\int_{\lambda_1}^{t}\theta(t,\omega)\frac{d}{d\omega}\{\omega^{-k}A_\lambda{}^{k}(\omega)\}\,d\omega$$

$$=\int_{\lambda_1}^{t}\theta(t,\omega)\,\frac{d}{d\omega}\,(\omega^{-k}A_\lambda{}^{k}(\omega))\,d\omega.$$

As $\dot\theta(\omega,\omega)=0$, in view of Lemma 2,

$$\{\phi(t)\}^{-k}\mathcal{J}(t)\quad\epsilon\quad BV(h,\infty),$$

if

$$\int_{\omega}^{\infty}|d_t\,[\theta(t,\omega)\{\phi(t)\}^{-k}]|<K. \tag{3.12}$$

But, after integrating by parts k-times, we have

$$\theta(t,\omega)=\sum_{j=0}^{k-1}K_j\omega^{j}\,(\partial/\partial\omega)^{j}\,[\{\phi(t)-\phi(\omega)\}^{k}\psi(\omega)] \tag{3.13}$$

where the K 's are constants.

Thus, from (3.11), (3.12) and (3.13), we see that

$$\{\phi(t)\}^{-k}B_\phi{}^{k}\{\phi(t)\}\quad\epsilon\quad BV(h,\infty),$$

if, for $0\leqslant r\leqslant k$,

$$\int_{\omega}^{\infty}|d_t\sum_{r}\omega^{r}\{\phi(t)\}^{-k}\,(\partial/\partial\omega)^{r}\,\{[\phi(t)-\phi(\omega)]^{k}\psi(\omega)\}]|<K.$$

Now $(\partial/\partial\omega)^{r}\,[\{\phi(t)-\phi(\omega)\}^{k}\psi(\omega)]$ is a sum of the terms of the type

$$\psi^{(\nu)}(\omega)\,(\partial/\partial\omega)^{r-\nu}\{\phi(t)-\phi(\omega)\}^{k}\qquad(0\leqslant\nu\leqslant r),$$

and $(\partial/\partial\omega)^{r-\nu}\{\phi(t)-\phi(\omega)\}^{k}$ is a linear combination of the terms of the form

$$\{\phi(t)-\phi(\omega)\}^{k-\mu}\prod_{p=1}^{r-\nu}\{\phi^{(p)}(\omega)\}^{\alpha_p}$$

where $0\leqslant\mu\leqslant r-\nu$,

$$\sum_{p=1}^{r-\nu}\alpha_p=\mu\qquad\text{and}\qquad\sum_{p=1}^{r-\nu}p\alpha_p=r-\nu.$$

To prove the theorem it is enough to show that

$$\int_{\omega}^{\infty}|d_t[\{\phi(t)\}^{-k}\omega^{r}\psi^{(\nu)}(\omega)\{\phi(t)-\phi(\omega)\}^{k-\mu}\prod_{p=1}^{r-\nu}\{\phi^{(p)}(\omega)\}^{\alpha_p}|<K.$$

But

$$\omega^r \psi^{(\nu)}(\omega) \prod_{p=1}^{r-\nu} \{\phi^{(p)}(\omega)\}^{\alpha_p}$$

$$=\omega^r \psi^{(\nu)}(\omega) \{\gamma(\omega)\}^{-(r-\nu)} \prod_{p=1}^{r-\nu} [O\{\phi(\omega)\}]^{\alpha_p}$$

$$=\left\{\frac{\omega}{\gamma(\omega)}\right\}^r \left[\{\gamma(\omega)\}^\nu \psi^{(\nu)}(\omega)\right] \prod_{p=1}^{r-\nu} [O\{\phi(\omega)\}^{\alpha_p}]$$

$$=\left\{\frac{\omega}{\gamma(\omega)}\right\}^k \left[O\left\{\frac{\gamma(\omega)}{\omega}\right\}^k\right] \left[O\{\phi(\omega)\}^\mu\right]$$

$$=O\left[\{\phi(\omega)\}^\mu \left\{\frac{\gamma(\omega)}{\omega}\right\}^{k-r}\right].$$

Hence, we have to show that

$$\int_\omega^\infty |d_t [\{\phi(t)\}^{-k}\{\phi(t)-\phi(\omega)\}^{k-\mu}\{\phi(\omega)\}^\mu \left\{\frac{\gamma(\omega)}{\omega}\right\}^{k-r}]|<K,$$

or, what is the same thing, that the total variation of

$$\chi(t, \omega)=\{\phi(t)\}^{-k}\{\phi(t)-\phi(\omega)\}^{k-\mu}\{\phi(\omega)\}^\mu \left\{\frac{\gamma(\omega)}{\omega}\right\}^{k-r}$$

is less than some constant independent of ω, as t increases from $\omega(\omega>h)$ to ∞.

It is evident that, if $\mu=0$, $\chi(t, \omega) \in BV(h, \infty)$. But if $\mu \geqslant 1$, $\{\phi(t)\}^{-k} \times\{\phi(t)-\phi(\omega)\}^{k-\mu}\{\phi(\omega)\}^\mu$ is monotonic increasing till $\phi(t)$ becomes equal to $(k/\mu)\phi(\omega)$, after which it is monotonic decreasing, but never becomes negative. Hence the total variation of a term of $\chi(t, \omega)$, when $1 \leqslant \mu \leqslant k$, is some multiple of $\{\gamma(\omega)/\omega\}^{k-r}$ and therefore, by the hypothesis (i) on $\gamma(\omega)$, less than some constant independent of ω.

This completes the proof of the theorem.

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ON THE ABSOLUTE SUMMABILITY FACTORS OF FOURIER SERIES

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1.1. Let Σa_n be a given infinite series. The series Σa_n is said to be absolutely summable (A), or summable $|A|$, if

$$F(x)=\Sigma a_n x^n$$

is convergent for $0\leqslant x<1$ and $F(x)$ is a function of bounded variation in the interval (0, 1), open on the right.[1]

Let $s_n^0=s_n$ denote the nth partial sum of Σa_n, and let s_n^k and t_n^k denote the nth Cesàro means of order $k\,(k>-1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series Σa_n is said to be absolutely summable (C, k), or summable $|C, k|$, if the sequence $\{s_n^k\}$ is of bounded variation, that is to say,

$$\Sigma\, |s_n^k-s_{n-1}^k|<\infty.^2$$

It is well-known that summability $|C, k|$ implies summability $|A|$.[3]

We shall require the following identities for $k>-1$.

(1.11) $$t_n^k=n(s_n^k-s_{n-1}^k),^4$$

(1.12) $$t_n^k=(A_n^k)^{-1}\sum_{\nu=1}^{n} A_{n-\nu}^{k-1}\,\nu a_\nu,$$

where

(1.13) $$A_n^k=\Gamma(n+k+1)/\Gamma(n+1)\,\Gamma(k+1)$$
$$\sim n^k/\Gamma(k+1).$$

We write $$\Delta u_n=u_n-u_{n+1},$$

and

$$\Delta^\sigma u_n=\sum_{\nu=n}^{\infty} A_{\nu-n}^{-\sigma-1}\,u_\nu\,,$$

1. Whittaker (19).
2. Fekete (9), Kogbetliantz (13).
3. Bosanquet (4), Fekete (11).
4. Kogbetliantz (13), (14).

provided the series on the right converges. In particular, when σ is a positive integer,

$$\Delta^\sigma u_n = \sum_{j=0}^{\sigma} (-1)^j \binom{\sigma}{j} u_{n+j}.$$

A sequence $\{\lambda_n\}$ is said to be *convex* if

$$\Delta^2 \lambda_n \geqslant 0.$$

We shall say that the sequence $\{\lambda_n\}$ is *hyper-convex of order h*, if

$$\Delta^{h+2} \lambda_n \geqslant 0, \quad (h=0, 1, \ldots).$$

By definition, hyper-convexity of order zero is the same as convexity.

1.2. Let $f(t)$ be a periodic function with period 2π, and integrable (L), that is, in the sense of Lebesgue, over $(-\pi, \pi)$. Without any loss of generality, we may assume that the constant term in the Fourier series of $f(t)$ is zero, that is,

(1.21) $$\int_{-\pi}^{\pi} f(t)\, dt = 0,$$

and

(1.22) $$f(t) \sim \Sigma (a_n \cos nt + b_n \sin nt) = \Sigma c_n(t).$$

We use the following notations.

$$\phi(t) = \phi_x(t) = (1/2)\{f(x+t) + f(x-t) - 2f(x)\};$$

(1.23) $$\begin{cases} \Phi_\alpha(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^1 (t-u)^{\alpha-1} \phi(u)\, du \quad (\alpha > 0), \\ \Phi_0(t) = \phi(t), \\ \phi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Phi_\alpha(t) \quad (\alpha \geqslant 0); \end{cases}$$

(1.24) $$K_n{}^\alpha(t) = (A_n{}^\alpha)^{-1} \Sigma A_{n-\nu}^{\alpha-1} \lambda_\nu \cos \nu t;$$

(1.25) $$\{F(t)\}_p = (\partial/\partial t)^p \{F(t)\}.$$

1.3. The absolute summability factors of Fourier series were first considered by Prasad, who proved the following.

THEOREM A.[1] If $\{\lambda_n\}$ be any one of the sequences

(1.31) $$\{(\log n)^{-1-\epsilon}\}, \ \{(\log n)^{-1} (\log \log n)^{-1-\epsilon}\},$$
$$\{(\log n)^{-1} (\log \log n)^{-1} \ldots (\log \log \ldots \log_{p-1} n)^{-1} (\log \log \ldots \log_p n)^{-1-\epsilon}\}, \quad (\epsilon > 0),$$

1. Prasad (18).

then $\Sigma\lambda_n c_n(t)$ is summable $|A|$ at every point $t=x$ where

$$\int_0^t |\phi(u)|\, du=o(t) \quad (t\to 0), \tag{1.32}$$

and, therefore, almost everywhere.

Theorem A was extended in the following form by Izumi and Kawata.[1]

THEOREM B. If $\{\lambda_n\}$ be a convex sequence such that $\Sigma n^{-1}\lambda_n$ is convergent, then $\Sigma \lambda_n c_n(t)$ is summable $|A|$ at every point $t=x$, where (1.32) holds.

In 1948 Cheng obtained a generalization of Theorem A in another direction by establishing

THEOREM C.[2] If $\{\lambda_n\}$ is any one of the sequences (1.31), and (1.32) holds, then $\Sigma\lambda_n c_n(x)$ is summable $|C, \alpha|$ for every $\alpha>1$.

In 1954, one of the authors[3] proved the following theorem, which includes as particular cases all these previous results.

THEOREM D. If $\{\lambda_n\}$ is a convex sequence such that $\Sigma n^{-1}\lambda_n$ is convergent, and (1.32) holds,[4] then $\Sigma\lambda_n c_n(x)$ is summable $|C, \alpha|$ for every $\alpha>1$.

The object of the present paper is to generalize Theorem D by replacing the condition (1.32) by the less stringent condition:

$$\int_0^t |\phi_h(u)|\, du=O(t) \quad (t\to 0), \tag{1.33}$$

and to establish a theorem, which includes as a particular case (for $h=0$) Theorem D.

2.1. We establish the following theorem.

THEOREM. *Let h be a positive integer $\geqslant 0$ and let $\{\lambda_n\}$ be a monotonic non-increasing sequence when $h=0$, and a hyper-convex sequence of order $(h-1)$ when $h\geqslant 1$, such that*

$$\begin{cases} \text{(i)} & \Sigma n^{-1}\lambda_n<\infty \\ \text{(ii)} & \Sigma n^h \Delta^{h+1}\lambda_n<\infty. \end{cases} \tag{2.11}$$

Then, if

$$\int_0^t |\phi_h(u)|\, du=O(t), \tag{2.12}$$

as $t\to 0$, $\Sigma\lambda_n c_n(x)$ is summable $|C, h+1+\delta|$ for every $\delta>0$.

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1. Izumi and Kawata (12).
2. Cheng (8).
3. Pati (15).
4. The condition (1.32) can, of course, be replaced by

$$\int_0^t |\phi(u)|du=O(t) \qquad (t\to 0).$$

2.2. We require a number of lemmas for the proof of our theorem.

LEMMA 1.[1] *Let* S_n^k *denote the nth Cesàro sum of order k* $(k \geqslant 0)$ *corresponding to the infinite series* $\sum_{1}^{\infty} (\sin nt)_{h+1}$ $(h \geqslant 0)$. *Then*

(i) $S_n^k = O(n^{k+h+2})$ *for* $0 < t \leqslant n^{-1}$;

(ii) $S_n^k = O(n^{h+1} t^{-k-1}) + O(n^k t^{-h-2})$ *for* $n^{-1} < t \leqslant \pi$.

LEMMA 2.[2] *If* $0 < k \leqslant 1$ *and* $1 \leqslant \nu < n$, *then*

$$\left| \sum_{\mu=1}^{\nu} A_{n-\mu}^{k-1} c_\mu \right| \leqslant \max_{1 \leqslant m \leqslant \nu} \left| \sum_{\mu=1}^{m} A_{m-\mu}^{k-1} c_\mu \right|.$$

LEMMA 3. (I) *If h is an integer* $\geqslant 1$ *and the sequence* $\{\lambda_n\}$ *is hyper-convex of order* $(h-1)$ *such that*

$$\text{(i)} \quad \Sigma n^{-1} \lambda_n < \infty$$

and

$$\text{(ii)} \quad \Sigma n^h \Delta^{h+1} \lambda_n < \infty,$$

then $\{\lambda_n\}$ *is hyper-convex of order* $(r-1)$, $r=1, 2, \ldots, h-1$, *and therefore* $\{\lambda_n\}$ *is a monotonic non-increasing positive sequence tending to zero.*

(II) *If* $\{\lambda_n\}$ *is a monotonic non-increasing sequence, such that* $\Sigma n^{-1} \lambda_n < \infty$, *then* $\{\lambda_n\}$ *is a positive sequence tending to zero.*

Proof. Since $\{\lambda_n\}$ is hyper-convex of order $(h-1)$, that is, $\Delta^{h+1} \lambda_n \geqslant 0$, (ii) is the same as

$$\Sigma n^h \, | \Delta^{h+1} \lambda_n | < \infty,$$

whence we infer that

$$\Sigma n^{h-1} | \Delta^h \lambda_n | < \infty. \tag{2.21}$$

Since (2.21) is a convergent series of positive terms,

$$n^h \, | \Delta^h \lambda_n | \to 0 \qquad (n \to \infty),$$

and therefore

$$| \Delta^h \lambda_n | \to 0 \qquad (n \to \infty),$$

whence $\Delta^h \lambda_n \to 0$.

But $\Delta^h \lambda_n$ is monotonic non-increasing. Hence, finally, $\Delta^h \lambda_n \geqslant 0$. By a repetition of this argument, we are ultimately led to the result that $\{\lambda_n\}$ is a convex sequence. Now, since (i) is satisfied, by virtue of Lemma 4 of Pati (15), (I) of the lemma is proved.

1. Pati (16).
2. Bosanquet (5), (6).

In order to prove part (II) we need only observe that, in view of the convergence of $\left\{ \sum_{\nu=1}^{n} \nu^{-1} \lambda_\nu \right\}$ (by (i)) and the result that if a sequence $\{s_n\}$ is convergent, then $\{n(s_n - s_{n+1})\}$ either oscillates or converges to zero, we infer that $\{\lambda_n\}$ can only tend to zero.

LEMMA 4. *If* $\alpha > h+1 \geqslant 1$, *and* (1.33) *holds, then*

$$\int_{n^{-1}}^{\pi} t^{h-\alpha} |\phi_h(t)|\, dt = O(n^{\alpha-h-1}),$$

as $n \to \infty$.

The proof requires only one integration by parts of $\int_{n^{-1}}^{\pi} t^{h-\alpha} |\phi_h(t)|\, dt$, and the use of the hypotheses.

LEMMA 5. *If* $\{\lambda_n\}$ *is hyper-convex of order* $(h-1)$, *when* $h \geqslant 1$, *or monotonic non-increaseing, when* $h=0$, *such that* $\Sigma n^{-1} \lambda_n < \infty$, *then*

$$\Sigma n^h \Delta^{h+1} \lambda_n < \infty \tag{2.22}$$

implies that

$$\Sigma (\log n) n^h \Delta^{h+1} \lambda_n < \infty.$$

Proof. We have

$$\Sigma = \sum_{1}^{m} (\log n)\, n^h \Delta^{h+1} \lambda_n$$

$$= \sum_{1}^{m-1} \Delta(n^h \log n) \sum_{\nu=1}^{n} \Delta^{h+1} \lambda_\nu + (m^h \log m) \sum_{1}^{m} \Delta^{h+1} \lambda_n$$

$$= \Delta^h \lambda_1 \left(\sum_{1}^{m-1} \Delta(n^h \log n) + m^h \log m\right)$$

$$- \sum_{1}^{m-1} \Delta(n^h \log n)\, \Delta^h \lambda_{n+1} - m^h \log m\, \Delta^h \lambda_{m+1}$$

$$= -\sum_{1}^{m-1} \Delta(n^h \log n)\, \Delta^h \lambda_{n+1} - m^h \log m\, \Delta^h \lambda_{m+1}$$

$$= O\left(\sum_{1}^{m-1} n^h |\Delta \log n|\, \Delta^h \lambda_{n+1}\right) + O\left(\sum_{1}^{m-1} \log(n+1)\, |\Delta n^h|\, \Delta^h \lambda_{n+1}\right)$$
$$+ O(m^h \log m\, \Delta^h \lambda_{m+1})$$

$$= O\left(\sum_{1}^{m-1} (n+1)^{h-1} \Delta^h \lambda_{n+1}\right) + O\left(\sum_{1}^{m-1} (n+1)^{h-1} \log(n+1)\, \Delta^h \lambda_{n+1}\right)$$
$$+ O(m^h \log m\, \Delta^h \lambda_{m+1})$$

$$= O(\Sigma_1) + O(\Sigma_2) + O(\Sigma_3), \text{ say.}$$

Now let us assume that

(2.23) $$\Sigma(\log n)n^{h-1}\Delta^h\lambda_n<\infty.$$

Then, obviously, $\Sigma_2=O(1)$. Also since (2.23) is a convergent series of positive terms, $\Sigma_3=O(1)$. Also, by virtue of (2.22), $\Sigma_1=O(1)$. Hence, finally, $\Sigma=O(1)$, if (2.23) holds. By successive application of such argument, we find that $\Sigma=O(1)$, if

(2.24) $$\Sigma\,(\log n)\,\Delta\lambda_n<\infty.$$

Now, by the conditions of this lemma, and an application of Lemma 3, $\{\lambda_n\}$ is a convex sequence, for $h\geqslant 1$, and a monotonic non-increasing sequence when $h=0$. Hence (2.24) holds, by virtue of a known result.[1]

LEMMA 6. *If $C_{n,\rho}{}^k$ denotes the nth Cesàro sum of order k corresponding to the series $\Sigma(-1)^n n^\rho$, then*

(i) $C_{n,\rho}{}^k=O(n^k)$ *for* $k\geqslant\rho$;

(ii) *when ρ is an even integer* $\geqslant 2$,

$$C_{n,\rho}{}^k=O(n^{\max(\rho,\,k-1)}).$$

Proof. (i) follows from the fact that $\Sigma(-1)^n n^\rho=O(1)$ (C,ρ). For (ii) see Pati (16).[2]

LEMMA 7.[3] *If $k\geqslant-1$, $r\geqslant 0$, necessary and sufficient conditions for* $\Sigma\epsilon_n a_n$ to *to be summable $|C, r|$ whenever*

$$s_n=a_1+\ldots\ldots+a_n=O(1)\ (C, k)$$

are

(i) $\Sigma n^{k-r}\,|\epsilon_n|<\infty$,

(ii) $\Sigma n^{-1}\,|\epsilon_n|<\infty$,

and (iii) $\Sigma n^k\,|\Delta^{k+1}\epsilon_n|<\infty$.

We shall require the use of this lemma in the special case when r and k are both positive integers and $r=k+1$. This case was considered originally by Fekete[4].

LEMMA 8. *For* $\alpha>r\geqslant 0$,

$$\sum_{n>\nu+1}(n-\nu)^{\alpha-r-1}\,n^{-1-\alpha}=O(\nu^{-r-1}).$$

The proof is easy.

1. Pati (15), Lemma 4.
2. Pati (16), Lemma 3(v).
3. Bosanquet and Chow (7), Theorem B.
4. Fekete (10).

3. *Proof of the theorem.* In view of the identity (1.11) and the consistency theorem, we need only prove that, for $h+1<\alpha<h+2$,

$$\Sigma n^{-1}|\zeta_n{}^\alpha|<\infty,$$

where

$$\zeta_n{}^\alpha=(2/\pi)\int_0^\pi \phi(t)\,K_n{}^\alpha(t)\,dt.$$

Integrating by parts h times, we have

$$\zeta_n{}^\alpha=(2/\pi)\left[\sum_{\rho=1}^{h}(-1)^{\rho-1}\Phi_\rho(t)\,(K_n{}^\alpha(t))_{\rho-1}\right]_0^\pi$$

$$+(-1)^h(2/\pi)\int_0^\pi \Phi_h(t)(K_n{}^\alpha(t))_h dt$$

$$=(2/\pi)J_1+(2/\pi)\{(-1)^h/\Gamma(h+1)\}\int_0^\pi \phi_h(t)\,\bar{K}_n{}^\alpha(t)\,dt, \text{ say,}$$

where

$$\bar{K}_n{}^\alpha(t)=t^h\,(K_n{}^\alpha(t))_h.$$

Hence, it is sufficient for our purpose to show that

$$\Sigma n^{-1}|J_1|<\infty \tag{3.1}$$

and

$$\Sigma n^{-1}\left|\int_0^\pi \phi_h(t)\,\bar{K}_n{}^\alpha(t)\right|<\infty. \tag{3.2}$$

Proof of (3.1).

Since $\sin n\pi=0$ and $\cos n\pi=(-1)^n$, it is evident from (1.11) and (1.12) that proving (3.1) is the same thing as proving the summability $|C,\alpha|$, $\alpha>h+1$, of

$$\Sigma(-1)^n\lambda_n n^{\rho-1}, \tag{3.3}$$

where ρ is an odd integer and $1\leqslant\rho\leqslant h$.

This follows by virtue of Lemmas 6 and 7.

Proof of (3.2). We have

$$\bar{K}_n{}^\alpha(t)=(t^h/A_n{}^\alpha)\sum_1^n A_{n-\nu}^{\alpha-1}\lambda_\nu\,(\sin\nu t)_{h+1}$$

$$=(t^h/A_n{}^\alpha)\,\Sigma, \text{ say.}$$

Now
$$\Sigma=\sum_1^{n-h-1}\Delta^{h+1}(A_{n-\nu}^{\alpha-1}\lambda_\nu)\,S_\nu{}^h$$

$$+\sum_{r=0}^{h}[\Delta^r(A_{n-\nu}^{\alpha-1}\lambda_\nu)]_{\nu=n-r}\,S_{n-r}{}^r$$

$$= \sum_{r=0}^{h+1} \binom{h+1}{r} \sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-r-1} \Delta^{h+1-r} \lambda_{\nu+r} S_\nu^h$$

$$+ \sum_{r=0}^{h} S_{n-r}{}^r \sum_{r'=0}^{r} \binom{r}{r-r'} A_r^{\alpha-1-(r-r')} \Delta^{r'} \lambda_{n-r'}$$

$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

Hence, we have to prove that

$$\Sigma n^{-1} \left| \int_0^\pi \phi_h(t)\, (t^h/A_n^\alpha) (\Sigma_1 + \Sigma_2)\, dt \right| < \infty,$$

and it suffices to show that

(3.4) $$\Sigma n^{-1-\alpha} \int_0^\pi |\phi_h(t)|\, t^h\, |\Sigma_1|\, dt < \infty$$

and

(3 5) $$\Sigma n^{-1-\alpha} \int_0^\pi |\phi_h(t)|\, |t^h|\, |\Sigma_2|\, dt < \infty.$$

Proof of (3.4).

It suffices to prove that, for $r=0, 1, \ldots, h$,

(3.6) $$\Sigma_{1,1} = \Sigma n^{-1-\alpha} \sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-r-1} \Delta^{h+1-r} \lambda_{\nu+r} \int_0^{\nu^{-1}} t^h |\phi_h(t)|\, |S_\nu^h|\, dt < \infty;$$

(3.7) for $r=0, 1, \ldots\ldots h$,

$$\Sigma_{1,2} = \Sigma n^{-1-\alpha} \sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-r-1} \Delta^{h+1-r} \lambda_{\nu+r} \int_{\nu^{-1}}^{\pi} t^h |\phi_h(t)|\, |S_\nu^h|\, dt < \infty;$$

and

(3.8) $$\Sigma_{1,3} = \Sigma n^{-1-\alpha} \int_0^\pi |\phi_h(t)| t^h \left| \sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-h-2} \lambda_{\nu+h+1} S_\nu^h \right| dt < \infty.$$

Proof of (3.6).

$$\Sigma_{1,1} \leqslant K \Sigma n^{-1-\alpha} \sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-r-1} \Delta^{h+1-r} \lambda_{\nu+r}\, \nu^{-h-1}\, \nu^{2h+2}$$

(by Lemma 1)

$$\leqslant K \Sigma n^{-1-\alpha} \sum_{\nu=1}^{n-h-1} \nu^{h+1} (n-\nu)^{\alpha-r-1} \Delta^{h+1-r} \lambda_{\nu+r}$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^{h+1} \Delta^{h+1-r} \lambda_\nu \sum_{n=\nu+h+1}^{\infty} (n-\nu)^{\alpha-r-1} n^{-1-\alpha}$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^{h-r} \Delta^{h-r+1} \lambda_\nu$$

$$\leqslant K < \infty,$$

by Lemmas 8 and 3.

Proof of (3.7).

$$\Sigma_{1,2} \leqslant K\Sigma n^{-1-\alpha} \sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-r-1} \Delta^{h+1-r}\lambda_{\nu+r} \times \int_{\nu^{-1}}^{\pi} t^h |\phi_h(t)| (\nu^{h+1}t^{-h-1}+\nu^h t^{-h-2})\, dt$$

$$\leqslant K\Sigma n^{-1-\alpha} \sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-r-1} \Delta^{h+1-r}\lambda_{\nu+r}\, \nu^{h+1}\int_{\nu^{-1}}^{\pi} t^{-1}|\phi_h(t)|\, dt$$

$$+ K\Sigma n^{-1-\alpha} \sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-r-1} \Delta^{h+1-r}\lambda_{\nu+r}\, \nu^{h}\int_{\nu^{-1}}^{\pi} t^{-2}|\phi_h(t)|\, dt$$

$$\leqslant K\Sigma n^{-1-\alpha} \sum_{\nu=1}^{n-h-1} (n-\nu)^{\alpha-r-1}\, \nu^{h+1}(\log \nu)\Delta^{h+1-r}\lambda_{\nu+r}$$

$$+ K\Sigma n^{-1-\alpha} \sum_{\nu=1}^{n-h-1} (n-\nu)^{\alpha-r-1}\, \nu^{h+1}\Delta^{h+1-r}\lambda_{\nu+r}$$

$$\leqslant K\sum_{\nu=1}^{\infty} (\log \nu)\, \nu^{h+1}\Delta^{h+1-r}\lambda_{\nu} \sum_{n=\nu+h+1}^{\infty} n^{-1-\alpha}(n-\nu)^{\alpha-r-1}$$

$$+ K\sum_{\nu=1}^{\infty} \nu^{h+1}\Delta^{h+1-r}\lambda_{\nu} \sum_{n=\nu+h+1}^{\infty} n^{-1-\alpha}(n-\nu)^{\alpha-r-1}$$

$$\leqslant K\sum_{\nu=1}^{\infty} (\log \nu)\,\nu^{h-r}\Delta^{h-r+1}\lambda_\nu + K\sum_{\nu=1}^{\infty} \nu^{h-r}\Delta^{h-r+1}\lambda_\nu$$

$$\leqslant K < \infty.$$

by Lemms 3 and 5.

Proof of (3.8).

We have

$$\sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-h-2} \lambda_{\nu+h+1} S_\nu^h = \sum_{\nu=1}^{n-h-2} \Delta\lambda_{\nu+h+1} \sum_{\mu=1}^{\nu} A_{n-\mu}^{\alpha-h-2} S_\mu^h + \lambda_n \sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-h-2} S_\nu^h + \lambda_n A_{h+1}^{\alpha-h-2} S_{n-h-1}^h$$

$$= O\Big(\sum_{\nu=1}^{n-h-2} \Delta\lambda_{\nu+h+1} \Big|\sum_{\mu=1}^{\nu} A_{n-\mu}^{a-h-2} S_\mu^h\Big|\Big) + O\Big(\lambda_n \Big|\sum_{\nu=1}^{n-h-1} A_{n-\nu}^{\alpha-h-2} S_\nu^h\Big|\Big) + O(\lambda_n |S_n^h|)$$

$$= O\left(\sum_{\nu=1}^{n-h-2} \Delta\lambda_{\nu+h+1} \max_{1\leqslant m\leqslant \nu} \Big|\sum_{\mu=1}^{m} A_{m-\mu}^{\alpha-h-2} S_\mu^h\Big|\right) \text{ (by Lemma 2)}$$

$$+ O(\lambda_n |S_n^{\alpha-1}|) + O(\lambda_n |S_n^h|)$$

$$=O\left(\sum_{\nu=1}^{n-h-2} \Delta\lambda_{\nu+h+1} \max_{1\leqslant m\leqslant \nu} |S_m^{\alpha-1}|\right)+O(\lambda_n|S_n^{\alpha-1}|)+O(\lambda_n|S_n^h|)$$

$$=O\left(\sum_{\nu=1}^{n-h-2} \Delta\lambda_{\nu+h+1}|S_\nu^{\alpha-1}|\right)+O(\lambda_n|S_n^{\alpha-1}|)+O(\lambda_n|S_n^h|) \text{ (since } \alpha>1).$$

Hence it suffices to show that

$$(3\cdot 9)\quad \begin{cases} \Sigma_{1,3,1}=\Sigma n^{-1-\alpha}\sum_{\nu=1}^{n-h-2}\Delta\lambda_{\nu+h+1}\int_0^{\nu^{-1}} t^h|\phi_h(t)|\,|S_\nu^{\alpha-1}|\,dt<\infty,\\ \Sigma_{1,3,2}=\Sigma n^{-1-\alpha}\sum_{\nu=1}^{n-h-2}\Delta\lambda_{\nu+h+1}\int_{\nu^{-1}}^{\pi} t^h|\phi_h(t)|\,|S_\nu^{\alpha-1}|\,dt<\infty,\\ \Sigma_{1,3,3}=\Sigma n^{-1-\alpha}\lambda_n\int_0^{n^{-1}} t^h|\phi_h(t)|\,|S_n^{\alpha-1}|\,dt<\infty,\\ \Sigma_{1,3,4}=\Sigma n^{-1-\alpha}\lambda_n\int_{n^{-1}}^{\pi} t^h|\phi_h(t)|\,|S_n^{\alpha-1}|\,dt<\infty,\\ \Sigma_{1,3,5}=\Sigma n^{-1-\alpha}\lambda_n\int_0^{n^{-1}} t^h|\phi_h(t)|\,|S_n^{h}|\,dt<\infty,\\ \Sigma_{1,3,6}=\Sigma n^{-1-\alpha}\lambda_n\int_{n^{-1}}^{\pi} t^h|\phi_h(t)|\,|S_n^{h}|\,dt<\infty. \end{cases}$$

Proof of (3·9).

$$\Sigma_{1,3,1}\leqslant K\Sigma n^{-1-\alpha}\sum_{\nu=1}^{n-h-2}\Delta\lambda_{\nu+h+1}\int_0^{\nu^{-1}} t^h|\phi_h(t)|\,\nu^{\alpha+h+1}\,dt$$

$$\leqslant K\Sigma n^{-1-\alpha}\sum_{\nu=1}^{n-h-2}\Delta\lambda_{\nu+h+1}\nu^{-h-1}\nu^{\alpha+h+1}$$

$$\leqslant K\sum_{\nu=1}^{\infty}\nu^\alpha\Delta\lambda_\nu\sum_{n=\nu+h+2}^{\infty}n^{-1-\alpha}$$

$$\leqslant K<\infty.$$

$$\Sigma_{1,3,2}\leqslant K\Sigma n^{-1-\alpha}\sum_{\nu=1}^{n-h-2}\Delta\lambda_{\nu+h+1}$$

$$\times\int_{\nu^{-1}}^{\pi} t^h|\phi_h(t)|\,(\nu^{h+1}t^{-\alpha}+\nu^{\alpha-1}t^{-h-2})\,dt$$

$$\leqslant K\Sigma n^{-1-\alpha}\sum_{\nu=1}^{n-h-2}\Delta\lambda_{\nu+h+1}\nu^{h+1}\int_{\nu^{-1}}^{\pi} t^{h-\alpha}|\phi_h(t)|\,dt$$

$$+K\Sigma n^{-1-\alpha}\sum_{\nu=1}^{n-h-2}\Delta\lambda_{\nu+h+1}\nu^{\alpha-1}\int_{\nu^{-1}}^{\pi} t^{-2}|\phi_h(t)|\,dt$$

$$\leqslant K\Sigma n^{-1-\alpha}\sum_{\nu=1}^{n-h-2}\Delta\lambda_{\nu+h+1}\,\nu^{\alpha}\quad\text{(by Lemma 4)}$$

$$+K\Sigma n^{-1-\alpha}\sum_{\nu=1}^{n-h-2}\Delta\lambda_{\nu+h+1}\,\nu^{\alpha}$$

$$\leqslant K\sum_{\nu=1}^{\infty}\nu^{\alpha}\Delta\lambda_{\nu}\sum_{n=\nu+h+1}^{\infty}n^{-1-\alpha}$$

$$\leqslant K<\infty.$$

$$\Sigma_{1,3,3}\leqslant K\Sigma n^{-\alpha-1}\lambda_n\int_0^{n^{-1}}t^h|\phi_h(t)|n^{\alpha+h+1}dt$$

$$\leqslant K\Sigma n^{-1}\lambda_n\leqslant K<\infty.$$

$$\Sigma_{1,3,4}\leqslant K\Sigma n^{-1-\alpha}\lambda_n\int_{n^{-1}}^{\pi}t^h|\phi_h(t)|\,(n^{h+1}t^{-\alpha}+n^{\alpha-1}t^{-h-2})dt$$

$$\leqslant K\Sigma\, n^{h-\alpha}\lambda_n\int_{n^{-1}}^{\pi}t^{h-\alpha}|\phi_h(t)|\,dt$$

$$+K\Sigma n^{-2}\lambda_n\int_{n^{-1}}^{\pi}t^{-2}|\phi_h(t)|dt$$

$$\leqslant K\Sigma n^{-1}\lambda_n+K\Sigma n^{-1}\lambda_n\quad\text{(by Lemma 4)}$$

$$\leqslant K<\infty.$$

$$\Sigma_{1,3,5}\leqslant Kn^{-1-\alpha}\lambda_n\int_0^{n^{-1}}t^h|\phi_h(t)|\,n^{2h+2}dt$$

$$\leqslant K\Sigma\lambda_n n^{2h-\alpha+1}n^{-h-1}$$

$$\leqslant K\Sigma n^{-1}\lambda_n n^{h+1-\alpha}$$

$$\leqslant K<\infty.$$

$$\Sigma_{1,3,6}\leqslant K\Sigma n^{-1-\alpha}\lambda_n\int_{n^{-1}}^{\pi}t^h|\phi_h(t)|\,(n^{h+1}t^{-h-1}+n^ht^{-h-2})dt$$

$$\leqslant K\Sigma n^{h-\alpha}\lambda_n\int_{n^{-1}}^{\pi}t^{-1}|\phi_h(t)|dt$$

$$+K\Sigma n^{h-1-\alpha}\lambda_n\int_{n^{-1}}^{\pi}t^{-2}|\phi_h(t)|dt$$

$$\leqslant K\Sigma n^{h-\alpha}(\log n)\lambda_n+K\Sigma n^{-1}\lambda_n n^{h+1-\alpha}\quad\text{(by Lemma 4)}$$

$$\leqslant K<\infty.$$

This completes the proof of (3.9), and hence of (3.4).

Proof of (3.5).

We have to show that

$$\Sigma n^{-1-\alpha}\int_0^\pi t^h|\phi_h(t)|\,\Big|\sum_{r=0}^{h} S_{n-r}{}^r \sum_{r'=0}^{r}\binom{r}{r-r'}A_r{}^{\alpha-1-(r-r')}\Delta^{r'}\lambda_{n-r'}\Big|dt<\infty.$$

It suffices to show the following.

For $r=0, 1, \ldots, (h-1)$; $r'=0, 1, \ldots, r$,

(3.10) $$\Sigma_r=\Sigma n^{-1-\alpha}\Delta^{r'}\lambda_{n-r'}\int_0^\pi t^h\,|\phi_h(t)|\,|S_{n-r}{}^r|\,dt<\infty,$$

and, for $r'=0, 1, \ldots, h$,

(3.11) $$\Sigma_h=\Sigma n^{-1-\alpha}\Delta^{r'}\lambda_{n-r'}\int_0^\pi t^h\,|\phi_h(t)|\,|S_{n-h}{}^h\,|\,dt<\infty.$$

Proof of (3.10).

$$\begin{aligned}\Sigma_r &\leqslant K\Sigma n^{-1-\alpha}\Delta^{r'}\lambda_{n-r'}\int_0^{n^{-1}} t^h\,|\phi_h(t)|\,n^{h+r+2}\,dt\\ &\quad+K\Sigma n^{-1-\alpha}\Delta^{r'}\lambda_{n-r'}\\ &\qquad\times\int_{n^{-1}}^{\pi} t^h\,|\phi_h(t)|\,(n^{h+1}t^{-r-1}+n^r t^{-h-2})\,dt\\ &\leqslant K\Sigma n^{h+r+1-\alpha}\Delta^{r'}\lambda_{n-r'}n^{-h-1}\\ &\quad+K\Sigma n^{h-\alpha}\Delta^{r'}\lambda_{n-r'}\int_{n^{-1}}^{\pi} t^{h-r-1}\,|\phi_h(t)|\,dt\\ &\quad+K\Sigma n^{r-\alpha-1}\Delta^{r'}\lambda_{n-r'}\int_{n^{-1}}^{\pi} t^{-2}\,|\phi_h(t)|\,dt\\ &\leqslant K\Sigma n^{r-\alpha}\Delta^{r'}\lambda_n+K\Sigma n^{h-\alpha}\Delta^{r'}\lambda_n+K\Sigma n^{r-\alpha}\Delta^{r'}\lambda_n\\ &\leqslant K<\infty,\end{aligned}$$

since $r\leqslant h-1$, $\alpha>h+1$.

Proof of (3.11).

$$\begin{aligned}\Sigma_h &\leqslant K\Sigma n^{-1-\alpha}\Delta^{r'}\lambda_n\int_0^{n^{-1}} t^h\,|\phi_h(t)|\,n^{2h+2}\,dt\\ &\quad+K\Sigma n^{-1-\alpha}\Delta^{r'}\lambda_n\int_{n^{-1}}^{\pi} t^h\,|\phi_h(t)|\,(n^{h+1}t^{-h-1}+n^h t^{-h-2})\,dt\\ &\leqslant K\Sigma n^{2h-\alpha+1}\Delta^{r'}\lambda_n n^{-h-1}\\ &\quad+K\Sigma n^{h-\alpha}\Delta^{r'}\lambda_n\int_{n^{-1}}^{\pi} t^{-1}\,|\phi_h(t)|\,dt\\ &\quad+K\Sigma n^{h-\alpha-1}\Delta^{r'}\lambda_n\int_{n^{-1}}^{\pi} t^{-2}\,|\phi_h(t)|\,dt\\ &\leqslant K\Sigma n^{h-\alpha}\Delta^{r'}\lambda_n+K\Sigma n^{h-\alpha}\log n\;\Delta^{r}\lambda_n+K\Sigma n^{h-\alpha}\Delta^{r'}\lambda_n\\ &\leqslant K<\infty,\end{aligned}$$

by Lemmas 3 and 5, since $-2<h-\alpha\leqslant r'-1\leqslant h-1$.

This completes the proof of (3.5). The theorem is thus established.

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